EXAMPLES OF RAUZY CLASSES

JEAN-CHRISTOPHE YOCCOZ

CONTENTS

Date: March 10, 2015.

The original paper is a preliminary version. Typo, minor corrections (in footnotes) and comments are due to C. Boissy.

1. General considerations and terminology

Let $\mathcal{A}$ be an alphabet with $d \geqslant 2$ letters. Let $\mathcal{R}$ be a Rauzy class on $\mathcal{A}$ and let $\mathcal{D}$ be the associated Rauzy diagram[1]. The involution $T \to T^{-1}$ for interval exchange maps corresponds to involutions of $\mathcal{A}$, $\mathcal{R}$, $\mathcal{D}$ which exchanges top and bottom[2].

Whenever possible, we will use $\mathcal{A} = \mathcal{A}_d$[3] in a way such that the involution on $\mathcal{A}$ is $m \mapsto -m$. In this case, we will always assume that

$$\pi_t(1-d) = 1 = \pi_b(d-1).$$

In other terms, we have ${}_t\alpha = 1-d$, ${}_b\alpha = d-1$. Recall that the letters[4] ${}_t\alpha, {}_b\alpha$ depend on $\mathcal{R}$, not on $\pi \in \mathcal{R}$.

Definition 1.1. A *pure cycle* of $\mathcal{D}$ is a cycle made of arrows of the same type (equivalently of the same name).

Definition 1.2. An element $\pi \in \mathcal{R}$ is *standard* if $\pi_t({}_b\alpha) = \pi_b({}_t\alpha) = d$. It is *semi-standard* of top (resp. bottom) type if one has $\pi_t({}_b\alpha) = d$ but $\pi_b({}_t\alpha) < d$ (resp. $\pi_b({}_t\alpha) < d$ but $\pi_t({}_b\alpha) = d$).

Definition 1.3. More generally, the *signature* of π is the pair $(d - \pi_b(\alpha_t), d - \pi_t(\alpha_b))$.

Summarizing, a vertex π is standard if its signature is $(d-1, d-1)$, semistandard if its signature is of the form $(d-1, j)$ or $(j, d-1)$, for some $j < d-1$. If π has signature (j, k), the length of the pure cycle of top type (resp. bottom type) through π is equal to j (resp. k).

Definition 1.4. A semistandard vertex π_1 is *attached* to a standard vertex π_0 if it belongs to one of the pure cycles through π_0. Such π_0 is unique. A vertex π which is neither standard nor semistandard is *linked* to a standard vertex π_0 if there exists a semistandard vertex π_1 attached to π_0 and a pure cycle through π_1 containing π.

A vertex π which is neither standard nor semistandard may be linked to 0, 1 or 2 standard vertices. Once π_0 is fixed, π_1 is uniquely determined.

Definition 1.5. A vertex π as above is *constrained* if it is linked to two standard vertices. It is *free* if it is not linked to any standard vertex. It is *open* if it is linked to exactly one standard vertex and is essential.

Definition 1.6. A vertex π is *inessential* if its signature has the form $(1, j)$ or $(j, 1)$ (Note that, except when $d = 2$, the signature cannot be equal to $(1, 1)$).

Remark 1.7. Let C be a pure cycle of top type length j. The signatures of the elements of C are distinct.

---

[1] For a general introduction on interval exchange maps and Rauzy classes, see for instance see J-C. Yoccoz, Interval exchange maps and translation surfaces. *Homogeneous flows, moduli spaces and arithmetic*, 1–69, Clay Math. Proc., 10, Amer. Math. Soc., Providence, RI, 2010.

[2] Fickensher proved that each Rauzy class contains a "self-inverse" element, *i.e.* an element invariant (up to a permutation I of the alphabet) when exchanging the top line and the bottom line. This corresponding permutation I of $\mathcal{A}$ is an involution, and the composition of the top/bottom exchange and I induces an involution of $\mathcal{R}$ and $\mathcal{D}$ (J. Fickensher: Self-inverses, Lagrangian permutations and minimal interval exchange transformations with many ergodic measures, *Commun. Contemp. Math.* 16 (2014)).

[3] $\mathcal{A}_d$ consists of the d integers in arithmetic progression $d-1,\ d-3, \ldots, 1-d$, see Section 3.

[4] ${}_t\alpha, {}_b\alpha$, the first letters of the top/bottom lines are to be distinguished from α_t, α_b, the last letters of the top/bottom lines. Note that J-C Yoccoz frequently uses $-\infty$ and $+\infty$ for ${}_t\alpha$ and ${}_b\alpha$.

Let π be a standard vertex. There are

- $(d-2)$ vertices of each type attached to π, more precisely one for each signature $(d-1, j)$ or $(j, d-1)$ $(1 \leqslant j \leqslant d-2)$;
- $(d-3)(d-2)$ vertices linked to π.

Therefore the total number of vertices related to π, including π itself, is equal to $1 + 2d - 4 + (d-2)(d-3) = (d-1)(d-2) + 1$.

One may define an unoriented graph[5] $\Gamma(\mathcal{D})$ whose vertices are the standard vertices of $\mathcal{D}$. For distinct standard vertices π, π', one has one edge joining π to π' as there are constrained vertices linked to π and π'.

Let us explain how to compute the edges of this graph. Let π be a standard vertex. For each pair (α, β) such that $\pi_t(\alpha) < \pi_t(\beta)$, $\pi_b(\alpha) < \pi_b(\beta)$ (in particular, $\Omega_{\alpha\,\beta} = 0$), there is an edge joining π to another standard vertex π' computed in the following way. If π reads as

$$\begin{pmatrix} {}_t\alpha & A & \alpha & B & \beta & C & {}_b\alpha \\ {}_b\alpha & X & \alpha & Y & \beta & Z & {}_t\alpha \end{pmatrix},$$

where A, B, C, X, Y, Z are words (which may be empty), then π' is equal to

$$\begin{pmatrix} {}_t\alpha & B & \beta & A & \alpha & C & {}_b\alpha \\ {}_b\alpha & Y & \beta & X & \alpha & Z & {}_t\alpha \end{pmatrix}.$$

Therefore, there are always 0 or 2 edges between two standard vertices. When there are two edges, the corresponding constrained vertices are

$$\begin{pmatrix} {}_t\alpha & A & \alpha & C & {}_b\alpha & B & \beta \\ {}_b\alpha & Y & \beta & Z & {}_t\alpha & X & \alpha \end{pmatrix}, \begin{pmatrix} {}_t\alpha & B & \beta & C & {}_b\alpha & A & \alpha \\ {}_b\alpha & X & \alpha & Z & {}_t\alpha & Y & \beta \end{pmatrix},$$

One can therefore omit the double edges in $\Gamma(\mathcal{D})$, as they are automatic!

Definition 1.8. The *default* $\delta(\pi)$ of a standard vertex π is the number of pairs (α, β) such that $\pi_t(\alpha) < \pi_t(\beta)$, $\pi_b(\alpha) < \pi_b(\beta)$. The number of zeros in Ω_π is equal to $d + 2\delta(\pi)$. The *default* $\delta(\mathcal{D})$ of the Rauzy diagram $\mathcal{D}$ is the number of edges (not counted twice) in $\Gamma(\mathcal{D})$. It is equal to

$$\delta(\mathcal{D}) = \frac{1}{2} \sum_{\pi} \delta(\pi),$$

where the sum is over the standard vertices of $\mathcal{D}$.

Definition 1.9. A pure cycle is *deep* if its length is > 1 and it does not contain any semi-standard vertex. A deep cycle is *hanging* if erasing its arrows disconnects the Rauzy diagram, *rooted* otherwise.

Definition 1.10. An automorphism[6] of $\mathcal{D}$ is a permutation σ of the alphabet $\mathcal{A}$ such that, for all $\pi \in \mathcal{R}$, the pair $(\pi_t \circ \sigma, \pi_b \circ \sigma)$ is also an element of $\mathcal{R}$[7].

[5] J. Fickenscher proved that $\Gamma(\mathcal{D})$ is always connected. See [A Combinatorial Proof of the Kontsevich-Zorich-Boissy Classification of Rauzy Classes, *Discrete and Continuous Dynamical Systems - Series A*, 2016], Proposition 5.1.

[6] Remark that a one-to-one map from $\mathcal{D}$ to $\mathcal{D}$ that send a "top" edge (*resp.* bottom) to a "top" edge (*resp.* bottom) is an automorphism.

[7] The computation of the order of the automorphism group can be found in [C. Boissy: Labeled Rauzy classes and framed translation surfaces. *Ann. Inst. Fourier (Grenoble)* 63 (2013)].

Remark 1.11. When a Rauzy diagram has no nontrivial automorphism, the top/bottom exchanging involution is uniquely defined. This is not always so in presence of non trivial automorphisms. Indeed, let I be such an involution, induced by an involution τ of $\mathcal{A}$, and let σ be a permutation of $\mathcal{A}$ inducing an automorphism of $\mathcal{D}$. If one has $\tau\sigma\tau = \sigma^{-1}$, then $\tau\sigma$ is an involution inducing a top/bottom exchanging involution[8] of $\mathcal{D}$.

1.1. **Height.** One defines the top and bottom heights $H_t(\pi), H_b(\pi)$ of a vertex (two even integers $\geqslant 0$) and the height $H(C)$ of a pure cycle (an odd integer > 0).

We write[9] $-\infty$ (resp. $+\infty$) for the first letter of the top (resp. bottom) lines of all the vertices of the diagram.

Let π be a vertex; denote as usual by α_t, α_b the last letters of the top and bottom lines of π. If π is a standard vertex, we set $H_t(\pi) = H_b(\pi) = 0$. We now assume that π is not a standard vertex.

We define $H_t(\pi)$ as follows. Let $d_t(1) := \pi_b(\alpha_t)$.

If $d_t(1) = 1$ (i.e if π is a semistandard vertex of top type) let $H_t(\pi) := 2$. Otherwise, define

$$d_t(2) := \min_{\pi_b(\alpha) > d_t(1)} \pi_t(\alpha).$$

If $d_t(2) = 1$, define $H_t(\pi) := 4$. Otherwise, define

$$d_t(3) := \min_{\pi_t(\alpha) > d_t(2)} \pi_b(\alpha).$$

If $d_t(3) = 1$, define $H_t(\pi) := 6$. Otherwise, define

$$d_t(4) := \min_{\pi_b(\alpha) > d_t(3)} \pi_t(\alpha).$$

We claim that the process must stop with some $d_t(k) = 1$, which corresponds to $H_t(\pi) = 2k$. Indeed, as π is irreducible, we must have $d_t(k+1) < d_t(k)$ as long as $d_t(k) > 1$. This proves the claim.

It is convenient to define $d_t(m)$ for all positive integers. If $H_t(\pi) = 2k$, we have $d_t(m) = 1$ for all $m \geqslant k$.

One defines similarly $H_b(\pi)$, starting with $d_b(1) := \pi_t(\alpha_b)$.

Proposition 1.12. *One has*

$$d_t(k+1) \leqslant d_b(k), \qquad d_b(k+1) \leqslant d_t(k), \qquad \forall k \geqslant 1$$

hence

$$|H_t(\pi) - H_b(\pi)| \leqslant 2.$$

Proof. This is clear by induction on k. □

Definition 1.13. The height $H(\pi)$ of a vertex π is

$$H(\pi) := \min(H_t(\pi), H_b(\pi)).$$

The height $H(C)$ of a pure cycle C is

$$H(C) := 1 + \min_{\pi \in C} H(\pi).$$

[8]Any top/bottom exchanging involution of $\mathcal{D}$ is obtained in this way since the composition of two such involutions is an automorphism. Numerical experiment suggests that any top/bottom exchanging involution of $\mathcal{D}$ fixes a vertex, although not necessarily a standard one.

[9]This notation is not always used.

Example 1.14. A vertex has height 0 iff it is standard, height 2 iff it is semistandard, height 4 iff it is linked to some standard vertex. A pure cycle has height 1 iff it contains a standard vertex, height 3 iff it contains a semistandard vertex but no standard vertex.

Proposition 1.15. *Let π be a non standard vertex. If $H_t(\pi) = 2k$ (resp. $H_b(\pi) = 2k$), then the pure cycle of top type (resp. of bottom type) through π has height $2k - 1$.*

Proof. We have to show that all vertices π' in the pure cycle C_t of top type through π have $H(\pi') \geqslant 2k - 2$, with at least one of them having $H(\pi') = 2k - 2$. Denote by $d'_t(m)$, $d'_b(m)$ the sequences defining $H_t(\pi'), H_b(\pi')$. It is clear that we have $d'_t(m) = d_t(m)$ for all $m \geqslant 1$, hence $H_t(\pi') = H_t(\pi)$ for all $\pi' \in C_t$. Therefore $H(\pi') \geqslant 2k - 2$. Let β be the letter such that $d_t(2) = \pi_t(\beta)$. By definition of $d_t(2)$, there is a vertex $\pi' \in C_t$ such that the last letter of the bottom line is β. For this vertex, we have $d'_b(m) = d_t(m + 1)$ for $m \geqslant 1$, hence $H(\pi') = H_b(\pi') = H_t(\pi) - 2$.

□

Corollary 1.16. *Let C be a pure cycle of top type and height $2k - 1 \geqslant 3$. All vertices $\pi \in C$ satisfy $H_t(\pi) = 2k$, hence $H(\pi) = 2k$ or $2k - 2$, with at least one satisfying $H(\pi) = H_b(\pi) = 2k - 2$.*

Corollary 1.17. *Let V be a vertex such that*[10] *$H_t(V) = 2k \geqslant 2$. There exists a finite sequence $(V_0, C_1, V_2, \dots, C_{2k-1}, V_{2k} = V)$ such that*

- *for $0 \leqslant i \leqslant k$, V_i is a vertex of height $2i$;*
- *for $0 < i \leqslant k$, C_{2i-1} is a pure cycle of height $2i - 1$;*
- *for $0 < i \leqslant k$, C_{2i-1} contains V_{2i-2} and V_{2i};*
- *C_{2k-1} is of top type.*

Proof. By induction on k. The case $k = 1$ is clear. Assume that $k > 1$ and that the conclusion of the corollary holds for $k - 1$. Let $V = V_{2k}$ as in the corollary. Let C_{2k-1} be the cycle of top type through V. By the proposition, the height of C_{2k-1} is equal to $2k - 1$. By the corollary above, C_{2k-1} contains a vertex $V' = V_{2k-2}$ with $H_t(V') = 2k, H_b(V') = 2k - 2$. We apply the induction hypothesis and get the required conclusion.

□

Proposition 1.18. *Let V be a vertex such that $H_t(V) = 4, H_b(V) = 6$. Let $V_0, C_1, V_2, C_3, V_4 = V$) as in the last corollary. Let α_1, α_2 be the letters such that $\pi_t(\alpha_1) = \pi_b(\alpha_2) = d$. Then V is inessential iff one has $\pi_t(\alpha_1) = \pi_t(\alpha_2) + 1$ in V_0.*

Proof. Clear □

1.2. **Chains.**

Definition 1.19. A *bimonotonous chain*[11] is a sequence $(V_0, C_1, \dots, C_{2k-1}, V_{2k})$ such that

- for $0 \leqslant 2i \leqslant k$, the height of the vertices V_{2i} and V_{2k-2i} is equal to $2i$;
- for $0 \leqslant 2i < k$, the height of the pure cycles C_{1+2i} and $C_{2k-1-2i}$ is equal to $2i + 1$;
- for $0 \leqslant 2i < 2k$, the pure cycle C_{1+2i} contains V_{2i} and V_{2i+2};
- the vertices $V_0, \dots, V_{2k}$ are distinct;
- the cycles $C_1, \dots C_{2k-1}$ are distinct, with alternating types.

[10] In the original version, Yoccoz mentions that the corollary must be reformulated since one can have $V = V'$ in the proof. Adding the hypothesis $H(V) = H_t(V)$ seems to solve this case.

[11] denoted simply *monotonous chain* or *monotonous cycle* in the remaining of the paper

The *length* of a monotonous cycle is the number k of pure cycles. It is at least equal to 4.

Remark 1.20. Let $(V_0, C_1, \dots, C_{2k-1}, V_{2k})$ be a monotonous cycle. Then $(V_{2k}, C_{2k-1}, \dots, C_1, V_0)$ is also a monotonous cycle, called the opposite cycle.

We will now analyze the monotonous chains of small lengths. In general, we denote by α_i the winner of C_{1+2i}. We have $\alpha_0 = \pm\infty$ according to the type of C_1 and similarly for α_{k-1}, hence only $\alpha_1, \dots, \alpha_{k-2}$ are really relevant. We will generally assume, unless stated otherwise, that C_1 is of top type.

1.2.1. *Monotonous chains of length* 4. This has been considered earlier. Such a chain is determined by a pair (α_1, α_2) such that we have in V_0

$$\pi_t(\alpha_1) < \pi_t(\alpha_2), \quad \pi_b(\alpha_1) < \pi_b(\alpha_2).$$

We have then[12]

$$V_0 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_1 \nearrow \alpha_2]_t & (\alpha_2 \nearrow +\infty]_t \\ \infty & (\infty \nearrow \alpha_1]_b & (\alpha_1 \nearrow \alpha_2]_b & (\alpha_2 \nearrow -\infty]_b \end{pmatrix},$$

$$V_2 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_1 \nearrow \alpha_2]_t & (\alpha_2 \nearrow +\infty]_t \\ \infty & (\alpha_1 \nearrow \alpha_2]_b & (\alpha_2 \nearrow -\infty]_b & (\infty \nearrow \alpha_1]_b \end{pmatrix},$$

$$V_4 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_2]_t \\ \infty & (\alpha_1 \nearrow \alpha_2]_b & (\alpha_2 \nearrow -\infty]_b & (\infty \nearrow \alpha_1]_b \end{pmatrix},$$

$$V_6 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_2]_t \\ \infty & (\alpha_1 \nearrow \alpha_2]_b & (\infty \nearrow \alpha_1]_b & (\alpha_2 \nearrow -\infty]_b \end{pmatrix},$$

$$V_8 = \begin{pmatrix} -\infty & (\alpha_1 \nearrow \alpha_2]_t & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t \\ \infty & (\alpha_1 \nearrow \alpha_2]_b & (\infty \nearrow \alpha_1]_b & (\alpha_2 \nearrow -\infty]_b \end{pmatrix}.$$

In V_8 (i.e, for the opposite chain), the condition on α_1, α_2 is now

$$\pi_t(\alpha_1) > \pi_t(\alpha_2), \quad \pi_b(\alpha_1) > \pi_b(\alpha_2).$$

1.2.2. *Monotonous chains of length* 5. Let $(V_0, C_1, \dots, C_9, V_{10})$ be a monotonous chain of length 5. We use the orders induced by V_0. One has

$$V_0 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_1 \nearrow +\infty]_t \\ +\infty & (+\infty \nearrow \alpha_1]_b & (\alpha_1 \nearrow -\infty]_b \end{pmatrix},$$

$$V_2 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_1 \nearrow +\infty]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_1]_b \end{pmatrix}.$$

The winner α_2 of C_5 must belong to $(\alpha_1 \nearrow +\infty)_t$. Moreover, as C_5 has height 5, it does not contain any semistandard vertex, hence $\alpha_2 \in (+\infty \nearrow \alpha_1]_b$. Then we have

$$V_4 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_2]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_2 \nearrow \alpha_1]_b \end{pmatrix}.$$

The winner α_3 of C_7 must belong to $(\alpha_2 \nearrow \alpha_1)_b$. Moreover, as C_7 has height 3, α_3 must belong either to $(-\infty \nearrow \alpha_1]_t$ or to $(\alpha_2 \nearrow +\infty]_t$.

[12] For instance, $(\alpha_1 \nearrow \alpha_2]_t$ means $\pi_t^{-1}(i_1+1), \pi_t^{-1}(i_1+2), \dots, \pi_t^{-1}(i_2)$, for $i_1 = \pi_t(\alpha_1)$ and $i_2 = \pi_t(\alpha_2)$, see also a similar notation in Section 18.5

- Assume that $\alpha_3 \in (-\infty \nearrow \alpha_1]_t$. Then we have

$$V_6 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_3]_t & (\alpha_3 \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_2]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix},$$

$$V_8 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_3]_t & (\alpha_1 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix},$$

$$V_{10} = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_3]_t & (\alpha_1 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t \\ +\infty & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b & (\alpha_1 \nearrow -\infty]_b \end{pmatrix}.$$

- Assume that $\alpha_3 \in (\alpha_2 \nearrow +\infty]_t$. Then we have

$$V_6 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow \alpha_3]_t & (\alpha_3 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_2]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix},$$

$$V_8 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow \alpha_3]_t & (\alpha_1 \nearrow \alpha_2]_t & (\alpha_3 \nearrow +\infty]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix},$$

$$V_{10} = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow \alpha_3]_t & (\alpha_1 \nearrow \alpha_2]_t & (\alpha_3 \nearrow +\infty]_t \\ +\infty & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b & (\alpha_1 \nearrow -\infty]_b \end{pmatrix}.$$

The **model** for the first case is

$$V_0 = \begin{pmatrix} -\infty & \alpha_3 & \alpha_1 & \alpha_2 & +\infty \\ +\infty & \alpha_2 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix},$$

$$V_{10} = \begin{pmatrix} -\infty & \alpha_3 & \alpha_2 & \alpha_1 & +\infty \\ +\infty & \alpha_2 & \alpha_1 & \alpha_3 & -\infty \end{pmatrix}.$$

The model for the second case is

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_2 & \alpha_3 & +\infty \\ +\infty & \alpha_2 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix},$$

$$V_{10} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_2 & +\infty \\ +\infty & \alpha_2 & \alpha_1 & \alpha_3 & -\infty \end{pmatrix}.$$

We see that the two models are actually symmetric to each other: if a monotonous chain is of the first type, the opposite chain is of the second type.

Notice also that the two models correspond to the vertices A^+, A^- of the diagram[13] $[5,2](2)(0)$. The monotonous chain connecting these two vertices may be transformed[14] into the concatenation of two chains of length 4: the edges in $\Gamma(\mathcal{D})$ connecting A^+ to S and S to A^-.

[13] See Sections 2 and 5.2 for the definitions of $[5,2](2)(0)$, A^+ and A^-.

[14] The precise meaning of the word “transformed” is unclear to us.

1.2.3. *More on monotonous chains of length* 5. We analyze the chain from the central cycle C_5 which is of top type and height 5. In this cycle, π_t stays the same, with $\pi_t(\alpha_2) = d$. The ordering π_b is also determined up to α_2, with a residual cyclic ordering on the remaining letters.

We have $\pi_b(-\infty) < \pi_b(\alpha_2)$. Otherwise C_5 would contain a vertex of height 2.

The letters α_1, α_3 are distinct and satisfy

$$\pi_b(\alpha_i) > \pi_b(\alpha_2), \quad \pi_t(\alpha_i) < \pi_t(+\infty),$$

for $i = 1, 3$. The two models above correspond to $\pi_t(\alpha_1) < \pi_t(\alpha_3)$ and $\pi_t(\alpha_1) > \pi_t(\alpha_3)$.

Considering only arrows with winner in $\{\pm\infty, \alpha_1, \alpha_2, \alpha_3\}$ one has an embedding of the diagram $[5,2](2)(0)$ in a "neighborhood" of the monotonous chain of length 5.

1.2.4. *Monotonous chains of length* 6. Let $(V_0, C_1, \ldots, C_{11}, V_{12})$ be a monotonous chain of length 6.

The beginning of the discussion is the same as before. However, as C_7 has now height 5, we must have $\alpha_3 \in (\alpha_1 \nearrow \alpha_2)_t$, hence

$$V_6 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_3]_t & (\alpha_3 \nearrow \alpha_2]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix}.$$

The winner α_4 of C_9 belongs to $(\alpha_3 \nearrow \alpha_2)_t$. As C_9 has height 3, we must have $\alpha_4 \in (\alpha_1 \nearrow -\infty]_b$. This gives

$$V_8 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_3]_t & (\alpha_4 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_4]_t \\ +\infty & (\alpha_1 \nearrow \alpha_4]_b & (\alpha_4 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix},$$

$$V_{10} = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_3]_t & (\alpha_4 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_4]_t \\ +\infty & (\alpha_1 \nearrow \alpha_4]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b & (\alpha_4 \nearrow -\infty]_b \end{pmatrix},$$

$$V_{12} = \begin{pmatrix} -\infty & (\alpha_1 \nearrow \alpha_3]_t & (\alpha_4 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_4]_t & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t \\ +\infty & (\alpha_1 \nearrow \alpha_4]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b & (\alpha_4 \nearrow -\infty]_b \end{pmatrix}.$$

The model for this chain is

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_2 & \alpha_3 & \alpha_1 & \alpha_4 & -\infty \end{pmatrix},$$

$$V_{12} = \begin{pmatrix} -\infty & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_1 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_1 & \alpha_3 & -\infty \end{pmatrix}.$$

This model is symmetric with the passage to opposite chains, exchanging top and bottom (because the chain has even length), α_i and α_{5-i}. The vertices V_0, V_{12} in the model correspond to the vertices[15] B^+, B^- of the diagram $[6,3](4)\mathrm{odd}$. The chain of length 6 connecting B^+ and B^- can be replaced[16] by the two edges in $\Gamma(\mathcal{D})$ connecting S to B^+ and B^-. With our notations, recall that we have

$$S = \begin{pmatrix} -\infty & \alpha_4 & \alpha_1 & \alpha_3 & \alpha_2 & +\infty \\ +\infty & \alpha_1 & \alpha_4 & \alpha_2 & \alpha_3 & -\infty \end{pmatrix}.$$

[15] See Sections 2 and 6 for the definitions of $[6,3](4)\mathrm{odd}$, B^+ and B^-.

[16] The precise meaning of the word "replaced" is unclear to us.

1.2.5. *More on monotonous chains of length* 6. We analyze the chain from the central vertex V_6 of height 6.

One has $\pi_t(\alpha_2) = \pi_b(\alpha_3) = d$. As C_5, C_7 have length > 3, one also have

$$\pi_t(+\infty) < \pi_t(\alpha_3), \quad \pi_b(-\infty) < \pi_b(\alpha_2).$$

On the other hands, as V_4, V_8 have height 4, one has

$$\pi_t(+\infty) > \pi_t(\alpha_1), \quad \pi_b(\alpha_1) > \pi_b(\alpha_2), \quad \pi_b(-\infty) > \pi_t(\alpha_4), \quad \pi_b(\alpha_4) > \pi_b(\alpha_3).$$

This gives the model for V_6:

$$V_6 = \begin{pmatrix} -\infty & \alpha_1 & +\infty & \alpha_3 & \alpha_4 & \alpha_2 \\ +\infty & \alpha_4 & -\infty & \alpha_2 & \alpha_1 & \alpha_3 \end{pmatrix}.$$

1.2.6. *Monotonous chains of length* 7. Let $(V_0, C_1, \ldots, C_{13}, V_{14})$ be a monotonous chain of length 7.

The beginning of the discussion, in particular the formula for V_6, is the same as before.

The winner α_4 of C_9 still belongs to $(\alpha_3 \nearrow \alpha_2)_t$. But as C_9 has now height 5, we cannot have $\alpha_4 \in (\alpha_1 \nearrow -\infty]_b$. Actually, the condition that the pure cycle of bottom type C_7 has height 7 means that no vertex of this cycle has height 4, which is equivalent to

$$(\alpha_1 \nearrow -\infty]_b \cap (\alpha_3 \nearrow \alpha_2]_t = \emptyset.$$

We have to consider three cases:

(1) $\alpha_4 \in (+\infty \nearrow \alpha_2)_b$.

We have then

$$V_8 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_3]_t & (\alpha_4 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_4]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_4]_b & (\alpha_4 \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix}.$$

(2) $\alpha_4 \in (\alpha_3 \nearrow \alpha_1)_b$.

We have then

$$V_8 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_3]_t & (\alpha_4 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_4]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_4]_b & (\alpha_4 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_3]_b \end{pmatrix}.$$

(3) $\alpha_4 \in (\alpha_2 \nearrow \alpha_3)_b$.

We have then

$$V_8 = \begin{pmatrix} -\infty & (-\infty \nearrow \alpha_1]_t & (\alpha_2 \nearrow +\infty]_t & (\alpha_1 \nearrow \alpha_3]_t & (\alpha_4 \nearrow \alpha_2]_t & (\alpha_3 \nearrow \alpha_4]_t \\ +\infty & (\alpha_1 \nearrow -\infty]_b & (+\infty \nearrow \alpha_2]_b & (\alpha_3 \nearrow \alpha_1]_b & (\alpha_2 \nearrow \alpha_4]_b & (\alpha_4 \nearrow \alpha_3]_b \end{pmatrix}.$$

As C_{11} has height 3, the winner α_5 of this cycle must belong to $(-\infty \nearrow \alpha_1]_t \cup (\alpha_2 \nearrow +\infty]_t$. We consider separately two possibilities.

- $\alpha_5 = \alpha_1$. This can only happen in cases (1) and (2) above. In this case the two possible models will have $d = 6$.

  In case (1), we have

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix},$$

$$V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & -\infty \end{pmatrix}.$$

In case (2), we have

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_1 & -\infty \end{pmatrix},$$

$$V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_2 & \alpha_4 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix}.$$

These correspond to opposite chains in the diagram $[6,2](1)(0,1)$ between the vertices Q and S^-.

- $\alpha_5 \neq \alpha_1$. In this case the letters α_i, $1 \leqslant i \leqslant 5$ are all distinct and the model will have $d = 7$. There are apparently 12 (!) distinct models

(1)

$$V_0 = \begin{pmatrix} -\infty & \alpha_5 & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_4 & \alpha_5 & \alpha_2 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_1 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_1 & \alpha_3 & \alpha_5 & -\infty \end{pmatrix},$$

(2)

$$V_0 = \begin{pmatrix} -\infty & \alpha_5 & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_5 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_1 & +\infty \\ +\infty & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_5 & -\infty \end{pmatrix},$$

(3)

$$V_0 = \begin{pmatrix} -\infty & \alpha_5 & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_3 & \alpha_5 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_1 & +\infty \\ +\infty & \alpha_4 & \alpha_1 & \alpha_3 & \alpha_2 & \alpha_5 & -\infty \end{pmatrix},$$

(4)

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & \alpha_5 & +\infty \\ +\infty & \alpha_4 & \alpha_5 & \alpha_2 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_1 & \alpha_3 & \alpha_5 & -\infty \end{pmatrix},$$

(5)

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & \alpha_5 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_5 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_5 & -\infty \end{pmatrix},$$

(6)

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & \alpha_5 & +\infty \\ +\infty & \alpha_4 & \alpha_2 & \alpha_3 & \alpha_5 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_4 & \alpha_1 & \alpha_3 & \alpha_2 & \alpha_5 & -\infty \end{pmatrix},$$

(7)

$$V_0 = \begin{pmatrix} -\infty & \alpha_5 & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_2 & \alpha_5 & \alpha_3 & \alpha_4 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_1 & +\infty \\ +\infty & \alpha_2 & \alpha_4 & \alpha_3 & \alpha_1 & \alpha_5 & -\infty \end{pmatrix},$$

(8)

$$V_0 = \begin{pmatrix} -\infty & \alpha_5 & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_1 & +\infty \\ +\infty & \alpha_2 & \alpha_4 & \alpha_1 & \alpha_3 & \alpha_5 & -\infty \end{pmatrix},$$

(9)

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & \alpha_5 & +\infty \\ +\infty & \alpha_2 & \alpha_5 & \alpha_3 & \alpha_4 & \alpha_1 & -\infty \end{pmatrix}, \ V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_2 & \alpha_4 & \alpha_3 & \alpha_1 & \alpha_5 & -\infty \end{pmatrix},$$

(10)

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & \alpha_5 & +\infty \\ +\infty & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_1 & -\infty \end{pmatrix},\ V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_2 & \alpha_4 & \alpha_1 & \alpha_3 & \alpha_5 & -\infty \end{pmatrix},$$

(11)

$$V_0 = \begin{pmatrix} -\infty & \alpha_5 & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & +\infty \\ +\infty & \alpha_2 & \alpha_4 & \alpha_5 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix},\ V_{14} = \begin{pmatrix} -\infty & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & \alpha_1 & +\infty \\ +\infty & \alpha_2 & \alpha_1 & \alpha_4 & \alpha_3 & \alpha_5 & -\infty \end{pmatrix},$$

(12)

$$V_0 = \begin{pmatrix} -\infty & \alpha_1 & \alpha_3 & \alpha_4 & \alpha_2 & \alpha_5 & +\infty \\ +\infty & \alpha_2 & \alpha_4 & \alpha_5 & \alpha_3 & \alpha_1 & -\infty \end{pmatrix},\ V_{14} = \begin{pmatrix} -\infty & \alpha_1 & \alpha_5 & \alpha_3 & \alpha_2 & \alpha_4 & +\infty \\ +\infty & \alpha_2 & \alpha_1 & \alpha_4 & \alpha_3 & \alpha_5 & -\infty \end{pmatrix}.$$

Here

- (1) and (12) are opposite chains in the diagram $[7,3](1)(3)$;
- (4) and (11) are opposite chains in the diagram $[7,3](3)(1)$;
- (2) and (9) are opposite chains in the diagram $[7,3](4)(0)O$;
- (6) and (7) are opposite chains in the diagram $[7,3](4)(0)O$;
- (3) and (9) are opposite chains in the diagram $[7,2](1)(1,0^2)$;
- (5) and (8) are opposite chains in the diagram $[7,2](1)(1,0^2)$;

1.2.7. *More on monotonous chains of length* 7. We analyze the chain from the central cycle C_7 of bottom type, height 7, winner α_3. The ordering π_b is the same for all elements of the cycle, with last letter α_3. The top ordering π_t is well-defined up to α_3 and is a cyclic ordering for larger elements.

One has $\pi_t(+\infty) < \pi_t(\alpha_3)$. Also, as C_7 has height 7, no letter α satisfies both $\pi_t(\alpha) > \pi_t(\alpha_3)$ and $\pi_b(\alpha) < \pi_b(-\infty)$. One has $\pi_t(\alpha_2) > \pi_t(\alpha_3)$ and also $\pi_t(\alpha_4) > \pi_t(\alpha_3)$. One has also

$$\pi_b(\alpha_1) > \pi_b(\alpha_2), \pi_t(\alpha_1) < \pi_t(+\infty), \quad \pi_b(\alpha_5) > \pi_b(\alpha_4), \pi_t(\alpha_5) < \pi_t(+\infty).$$

The letters α_i are distinct except possibly $\alpha_1 = \alpha_5$.

1.2.8. *Monotonous chains of length* 8. We analyze the chain $(V_0, C_1, \dots, C_{15}, V_{16})$ from the central vertex V_8. The only possible model is

$$V_8 = \begin{pmatrix} -\infty & \alpha_1 & +\infty & \alpha_5 & \alpha_6 & \alpha_3 & \alpha_2 & \alpha_4 \\ +\infty & \alpha_6 & -\infty & \alpha_2 & \alpha_1 & \alpha_4 & \alpha_5 & \alpha_3 \end{pmatrix}.$$

This vertex belongs to a diagram $[8,4](6)$ which is not hyperelliptic[17].

For a monotonous chain of length 10 the model for V_{10} is

$$V_{10} = \begin{pmatrix} -\infty & \alpha_1 & +\infty & \alpha_7 & \alpha_8 & \alpha_3 & \alpha_2 & \alpha_5 & \alpha_6 & \alpha_4 \\ +\infty & \alpha_8 & -\infty & \alpha_2 & \alpha_1 & \alpha_6 & \alpha_7 & \alpha_4 & \alpha_3 & \alpha_5 \end{pmatrix}.$$

The stratum is $[10,4](2,2,2)$.

For a monotonous chain of length 12 the model for V_{12} is

$$V_{12} = \begin{pmatrix} -\infty & \alpha_1 & +\infty & \alpha_9 & \alpha_{10} & \alpha_3 & \alpha_2 & \alpha_7 & \alpha_8 & \alpha_5 & \alpha_4 & \alpha_6 \\ +\infty & \alpha_{10} & -\infty & \alpha_2 & \alpha_1 & \alpha_8 & \alpha_9 & \alpha_4 & \alpha_3 & \alpha_6 & \alpha_7 & \alpha_5 \end{pmatrix}.$$

The stratum is $[12,6](10)$, not hyperelliptic.

[17]In the original version, Yoccoz wrote as a comment: "computation seems to indicate the even component"

Considering only monotonous chains of even length ℓ, there is a periodicity of order 6: When $\ell = 6m+4$ (with $m \geqslant 0$) , the stratum for the model is $[6m+4, 3m+1](2m, 2m, 2m)$. When $\ell = 2m$ ($m \geqslant 1$), with $m+1 \neq 0 \mod .3$, the stratum is $[2m, m](2m-2)$. In all cases, one should compute the connected component!!

2. LISTS OF RAUZY DIAGRAMS FOR SMALL d

We omit below the hyperelliptic diagrams and the genus 1 diagrams, which take care of all diagrams for $d \leqslant 4$. The notation[18] is

$$[d, g](\kappa_0)(\kappa_1, \dots, \kappa_{s-1}).$$

Here, d is the size of the alphabet, g is the genus, κ_i are the orders of the zeros at the s marked points; κ_0 is the order of the zero at the marked point which is the root of the Rauzy-Veech algorithm. The other κ_i (if any) are arranged in nonincreasing order. If necessary, one adds a parity sign O for odd or E for even[19].

- $[5, 2](0)(2)$, $[5, 2](2)(0)$.
- $[6, 3](4)O$, $[6, 2](0)(1, 1)$, $[6, 2](1)(1, 0)$, $[6, 2](0)(2, 0)$, $[6, 2](2)(0, 0)$.

3. HYPERELLIPTIC CLASSES

This is copied from [AMY][20].

Let $d \geqslant 2$ be an integer. Let $\mathcal{A}_d$ be the alphabet whose d elements are the integers in arithmetic progression $d-1, d-3, \dots, 1-d$. Let ι be the involution $k \mapsto -k$ of $\mathcal{A}_d$. We define inductively the hyperelliptic Rauzy class $\mathcal{R}_d$ over $\mathcal{A}_d$ and the associated Rauzy diagram $\mathcal{D}_d$. The Rauzy class $\mathcal{R}_d$ contains a central vertex $\pi^* = \pi^*(d)$ defined by

$$\pi^*_t(k) = \frac{1}{2}(d+1+k), \quad \pi^*_b(k) = \frac{1}{2}(d+1-k).$$

For $d = 2$, this is the only vertex. For $d \geqslant 2$, $\mathcal{R}_{d+1}$ is the disjoint union of $\pi^*(d+1)$, $j_t(\mathcal{R}_d)$ and $j_b(\mathcal{R}_d)$, where the injective maps j_t, j_b are defined as follows: for $\pi \in \mathcal{R}_d$, writing $j_t(\pi) = t\pi$, $j_b(\pi) = b\pi$, we have

$$t\pi_t(-d) = 1, \qquad t\pi_b(-d) = \pi_b(d-3),$$
$$t\pi_t(k) = 1 + \pi_t(k-1),$$
$$t\pi_b(k) = \begin{cases} \pi_b(k-1) & \text{if } \pi_b(k-1) < \pi_b(d-3), \\ \pi_b(k-1)+1 & \text{if } \pi_b(k-1) \geqslant \pi_b(d-3), \end{cases}$$

for $2-d \leqslant k \leqslant d$, and

$$b\pi_b(d) = 1, \qquad b\pi_t(d) = \pi_t(3-d),$$
$$b\pi_b(k) = 1 + \pi_b(k+1),$$
$$b\pi_t(k) = \begin{cases} \pi_t(k+1) & \text{if } \pi_t(k+1) < \pi_t(3-d), \\ \pi_t(k+1)+1 & \text{if } \pi_b(k+1) \geqslant \pi_t(3-d), \end{cases}$$

for $-d \leqslant k \leqslant d-2$.

[18]J-C Yoccoz also frequently uses the notation $[d, g](\kappa_0)(\kappa_1^{n_1}, \dots, \kappa_{s-1}^{n_{s-1}})$, meaning that each κ_i appears n_i times.

[19]Note that a Rauzy class is uniquely defined, up to a change of alphabet, by the root κ_0, the stratum of the moduli space of Abelian differential (*i.e.* $\{\kappa_0, \dots, \kappa_{s-1}\}$), and the corresponding connected component of the stratum (hyperelliptic, odd or even spin structure).

[20]A. Avila, C. Matheus, J-C. Yoccoz: Zorich conjecture for hyperelliptic Rauzy-Veech groups. *Math. Ann.* 370 (2018)

The one-to-one maps R_t, R_b from $\mathcal{R}_d$ to itself determining the arrows of $\mathcal{D}_d$ verify

$$\begin{cases} R_t(\pi^*(d+1)) = j_t(\pi^*(d)), \\ R_b(\pi^*(d+1)) = j_b(\pi^*(d)), \end{cases}$$

$$\begin{cases} R_t \circ j_b \circ R_t^{-1} = j_b, \\ R_b \circ j_t \circ R_b^{-1} = j_t, \end{cases}$$

$$\begin{cases} R_t \circ j_t \circ R_t^{-1}(\pi) = j_t(\pi), & \pi \neq \pi^*(d), \\ R_b \circ j_b \circ R_b^{-1}(\pi) = j_b(\pi), & \pi \neq \pi^*(d), \end{cases}$$

$$R_t \circ j_t \circ R_t^{-1}(\pi^*(d)) = \pi^*(d+1) = R_b \circ j_b \circ R_b^{-1}(\pi^*(d)).$$

The involution I_d on $\mathcal{R}_d$ defined by

$$I_d((\pi_t, \pi_b)) := (\pi_b \circ \iota, \pi_t \circ \iota)$$

satisfies

$$I_d(\pi^*(d)) = \pi^*(d), \quad I_{d+1} \circ j_b \circ I_d = j_t,$$

$$I_d \circ R_b \circ I_d = R_t.$$

Remark 3.1. There is a natural one-to-one correspondence W_d from the elements of $\mathcal{R}_d$ to the words in $\{t, b\}$ of length $< d-1$: namely, $W_d(\pi^*(d))$ is the empty word, $W_d(j_t(\pi))$ is the word $tW_{d-1}(\pi)$ and $W_d(j_b(\pi))$ is the word $bW_{d-1}(\pi)$. The involution I_d corresponds to the exchange of the letters t, b. One has also

$$W_d(R_t(\pi)) = W_d(\pi)t, \quad W_d(R_b(\pi)) = W_d(\pi)b, \quad \text{if } |W_d(\pi)| < d-2.$$

When $|W_d(\pi)| = d-2$, one writes $W_d(\pi) = W't^m$ with $m \geqslant 0$ and W' empty or finishing by b; one has then $W_d(R_t(\pi)) = W'$. Similarly for $W_d(R_b(\pi))$.

It is also not difficult to recover from $W_d(\pi)$ the winners of the arrows starting from π: the winner of the arrow of top type starting from π is the letter $d-1-2w_b(\pi)$ of $\mathcal{A}_d$, where $w_b(\pi)$ is the number of occurrences of b in $W_d(\pi)$; similarly, the winner of the arrow of bottom type starting from π is the letter $1-d+2w_t(\pi)$ of $\mathcal{A}_d$. Observe that we have always

$$d-1-2w_b(\pi) > 1-d+2w_t(\pi).$$

We now state another property of the hyperelliptic Rauzy diagrams which will be useful: given any vertex $\pi \in \mathcal{R}_d$, there is a **unique** oriented **simple** path in $\mathcal{D}_d$ from $\pi^*(d)$ to π. (A path is *simple* if it does not pass more than once through any vertex). Indeed, this is best seen through the representation of the vertices given in the remark above: the length of such a path is $|W_d(\pi)|$ and the path itself is through the sequence of initial subwords of $W_d(\pi)$. We will denote by $\gamma^*(\pi)$ this path.

Observe that all **simple** loops of positive length in $\mathcal{R}_d$ are *elementary*, i.e made of arrows of the same type (and consequently with the same winner). For any such loop γ, there is a unique vertex π such that γ passes through π but $\gamma^*(\pi)$ does not contain any arrow of γ; one checks that π is the vertex of γ such that $|W_d(\pi)|$ is minimal. One has

$$|\gamma| + |W_d(\pi)| = d-1.$$

The hyperelliptic diagrams have no non trivial automorphisms.

4. GENUS 1 DIAGRAMS

There is one genus one diagram for each $d \geqslant 2$. For $d = 2$ and $d = 3$, they are also hyperelliptic. To describe these diagrams, we choose an alphabet with two special letters $\aleph = {}_t\alpha$ and $\daleth = {}_b\alpha$. We denote by $\mathcal{A}^*$ the subset formed by the other $d-2$ letters.

The standard vertices are in one-to-one correspondence with the bijections between $\mathcal{A}^*$ and $\{2, \dots, d-1\}$: for standard vertices π_t and π_b coincide on $\mathcal{A}^*$.

Let π be a standard vertex. The default takes its maximal value $1/2(d-2)(d-3)$. All vertices which are linked to π are constrained. Therefore there is no free vertex, nor deep cycle, in $\mathcal{D}$. The edges of $\Gamma(\mathcal{D})$ from π are in one-to-one correspondence with pairs of integers (a, b) with $2 \leqslant a < b \leqslant d-1$. The other extremity of the (a, b)-edge is the vertex π' satisfying, for $\alpha \in \mathcal{A}^*$

$$\pi'(\alpha) = \begin{cases} \pi(\alpha) + b - a & \text{if} \quad 2 \leqslant \pi(\alpha) \leqslant a, \\ \pi(\alpha) - a + 1 & \text{if} \quad a < \pi(\alpha) \leqslant b, \\ \pi(\alpha) & \text{if} \quad b < \pi(\alpha) \leqslant d-1. \end{cases}$$

The total number of standard vertices is $(d-2)!$. The default of $\mathcal{D}$ is

$$\delta(\mathcal{D}) = \frac{1}{4}(d-2)(d-3)(d-2)!.$$

The symmetry group of the diagram is the symmetric group of $\mathcal{A}^*$.

The involution of $\mathcal{D}$ exchanges $\aleph$ and $\daleth$ and fixes each letter in $\mathcal{A}^*$. It fixes every standard vertex of $\mathcal{D}$. For every pair of standard vertices which are the extremities of an edge of $\Gamma(\mathcal{D})$, the involution exchanges the two vertices which are linked to both of them.

The genus 1 diagrams have large groups of automorphisms, isomorphic to the group of permutations of $\mathcal{A}^*$.

The total number of vertices is

$$N(\mathcal{D}) = \frac{1}{2}d!.$$

5. THE TWO DIAGRAMS WITH $d = 5, g = 2$ AND A DOUBLE ZERO

5.1. **The diagram** $[5, 2](0), (2)$**.** The automorphism group of this diagram is cyclic of order 3. Instead of a canonical involution, there are three of them. We choose as alphabet $-1 = {}_t\alpha, 1 = {}_b\alpha, a, b, c$.

There are three standard vertices

$$S_c := \begin{pmatrix} -1 & a & b & c & 1 \\ 1 & b & a & c & -1 \end{pmatrix}$$

and the two others deduced by cyclic permutation of a, b, c. Each of the three involution fixes one standard vertex and exchanges the other two. The involution I_c fixing S_c exchanges the letters -1 and 1, a and b, and fixes c. It has two other fixed points, which are the two constrained vertices linked to S_a and S_b.

The diagram $\Gamma(\mathcal{D})$ is the full graph on 3 vertices.

The default of each standard vertex is equal to 2, hence the default of the diagram is equal to 3.

Of the 3 pairs of symmetric vertices linked to a standard vertex, 1 is inessential and the other two are constrained. Therefore there are neither free vertices nor deep cycles. The total number of constrained vertices is equal to 6.

The total number of vertices is equal to

$$N(\mathcal{D}) = 3 \times [(d-1)(d-2)+1] - 6 = 33.$$

5.2. **The diagram** $[5,2](2),(0)$**.** We use as alphabet $\mathcal{A}_5$. The involution is $j \mapsto -j$. There are three standard vertices. One is the unique fixed point of the involution:

$$S := \begin{pmatrix} -4 & 0 & -2 & 2 & 4 \\ 4 & 0 & 2 & -2 & -4 \end{pmatrix}.$$

The other two form a symmetric pair

$$A^+ := \begin{pmatrix} -4 & -2 & 0 & 2 & 4 \\ 4 & 2 & -2 & 0 & -4 \end{pmatrix}, A^- := \begin{pmatrix} -4 & -2 & 2 & 0 & 4 \\ 4 & 2 & 0 & -2 & -4 \end{pmatrix}.$$

The defaults are $\delta(S) = 2, \delta(A^+) = \delta(A^-) = 1$. The default of the diagram is equal to 2.

- Of the 6 vertices linked to S, 2 are inessential and 4 are constrained; 2 of them are linked to A^+ and 2 to A^-.
- Of the 6 vertices linked to A^+ (or A^-), 2 are inessential and 2 are constrained, linked to S.

There are 2 deep cycles symmetric to each other. Each has length 2 and their vertices are linked to (A^+, A^-).

The total number of constrained vertices is equal to 4. The total number of vertices is

$$N(\mathcal{D}) = 3 \times [(d-1)(d-2)+1] - 4 = 35.$$

This Rauzy diagram has no nontrivial automorphism.

5.3. **More on the diagram** $[5,2](2),(0)$**.** We change the alphabet to $\mathcal{A} = \{\pm\infty, \pm 1, 0\}$ so that

$$S := \begin{pmatrix} -\infty & 0 & -1 & 1 & +\infty \\ +\infty & 0 & 1 & -1 & -\infty \end{pmatrix}.$$

Remark 5.1. It is also better to rename A^+ as $A(-1)$ and A^- as $A(1)$ but we don't do that at the moment.

The diagram is essentially made of

- 4 monotonous chains of length 4. Two chains connect the top cycle through A^+/A^- to the bottom cycle through S. The last two pure cycles in these chains are the same. The other two chains connect the bottom cycle through A^+/A^- to the top cycle through S, and are the images of the first two chains by the involution.
- 2 monotonous chains of length 5. They connect the top (resp. bottom) cycles through A^+ and A^-.

We have omitted in this description the pure cycles of length 1.

We compute the winners of the pure cycles of length > 1.

The winner of a pure cycle of top type through a standard vertex is $+\infty$. The winner of a pure cycle of bottom type through a standard vertex is $-\infty$.

The monotonous chains of length 4 from the bottom cycle through S to the top cycles through A^+ (resp. A^-) have non trivial successive winners 0 (for both) and -1 (resp. $+1$).

The monotonous chain of length 5 from the top cycle through A^+ to the top cycle through A^- has non trivial successive winners $0, 1, -1$.

The winners in the other chains are obtained from the involution.

6. The diagram $[6,3](4)$ Odd

6.1. **Standard vertices.** We use the alphabet $\{\pm\infty, \pm 2, \pm 1\}$. There are no nontrivial automorphism. The canonical involution sends k to $-k$.

There are 7 standard vertices. One is fixed by the canonical involution of $\mathcal{D}$

$$S := \begin{pmatrix} -\infty & -2 & 2 & -1 & 1 & \infty \\ \infty & 2 & -2 & 1 & -1 & -\infty \end{pmatrix}.$$

The others 6 come into 3 pairs of symmetric vertices

$$A^+ := \begin{pmatrix} -\infty & 2 & -1 & 1 & -2 & \infty \\ \infty & 1 & 2 & -2 & -1 & -\infty \end{pmatrix}, \quad A^- := \begin{pmatrix} -\infty & -1 & -2 & 2 & 1 & \infty \\ \infty & -2 & 1 & -1 & 2 & -\infty \end{pmatrix},$$

$$B^+ := \begin{pmatrix} -\infty & 2 & -1 & -2 & 1 & \infty \\ \infty & 1 & -1 & 2 & -2 & -\infty \end{pmatrix}, \quad B^- := \begin{pmatrix} -\infty & -1 & 1 & -2 & 2 & \infty \\ \infty & -2 & 1 & 2 & -1 & -\infty \end{pmatrix},$$

$$C^+ := \begin{pmatrix} -\infty & 2 & 1 & -1 & -2 & \infty \\ \infty & 1 & -2 & -1 & 2 & -\infty \end{pmatrix}, \quad C^- := \begin{pmatrix} -\infty & -1 & 2 & 1 & -2 & \infty \\ \infty & -2 & -1 & 1 & 2 & -\infty \end{pmatrix}.$$

The involution has another fixed point, which is essential.

$$F := \begin{pmatrix} -\infty & 2 & \infty & -1 & -2 & 1 \\ \infty & -2 & -\infty & 1 & 2 & -1 \end{pmatrix}.$$

It is free of signature $(2,2)$. There are 4 other free vertices, of signature $(3,1)$, $(1,3)$, $(1,2)$, $(2,1)$ respectively, which are inessential and form two symmetric pairs with respect to the involution.

- Of the 12 vertices linked to S, 4 are inessential and 8 are constrained; of these, 2 are linked to A^+, 2 to A^-, 2 to B^+ and two to B^-. One has $\delta(S) = 4$.
- Of the 12 vertices linked to A^-, 3 are inessential, 6 are constrained and 3 are neither constrained nor inessential. Among the 6 constrained vertices, 2 are linked to S, 2 to C^+ and 2 to B^+. One has $\delta(A^+) = \delta(A^-) = 3$.
- Of the 12 vertices linked to B^+, 4 are inessential, 4 are constrained and 4 are neither constrained nor inessential. Among the 4 constrained vertices, 2 are linked to S and 2 to A^-. One has $\delta(B^+) = \delta(B^-) = 2$.
- Of the 12 vertices linked to C^+, 4 are inessential, 4 are constrained and 4 are neither constrained nor inessential. Among the 4 constrained vertices, 2 are linked to A^- and 2 to C^-. One has $\delta(C^+) = \delta(C^-) = 2$.
- The statistics for A^+, B^-, C^- are deduced from the involution.

The total number of constrained vertices is thus equal to 18. The total number of vertices is equal to

$$N(\mathcal{D}) = 7 \times [(d-1)(d-2)+1] - 18 + 5 = 134.$$

The default $\delta(\mathcal{D})$ of $\mathcal{D}$ is equal to 9.

6.2. **Deep cycles.** There are 6 pairs of symmetric deep cycles. Only one of them is hanging. Of the 5 pairs of deep cycles which are rooted:

- 2 pairs have length 2, containing two linked vertices, respectively to $(A^-, C^-), (A^+, C^+), (B^+, A^+), (B^-, A^-)$.
- one pair has length 3, containing one free inessential vertex and two linked vertices, respectively to (B^+, C^+) and (B^-, C^-)[21].
- one pair has length 3, containing three linked vertices to (A^-, B^-, C^-) and (A^+, B^+, C^+).
- The last two symmetric cycles are attached at the fixed point F of the involution. They have length 2, the other vertex is linked to B^+/B^-.

This Rauzy diagram has no non trivial automorphism.

6.3. **Analysis by increasing height.** One has 7 standard vertices of height 0. Each produces 2 pure cycles of height 1, 14 in total.

Each pure cycle of height 1 contains one standard vertex and 4 vertices of height 2. Therefore there are 56 vertices of height 2.

Through each vertex of height 2, there is one pure cycle of height 1 and one pure cycle of height 3. Therefore there are 56 pure cycles of height 3. They have length $1, 2, 3$ or 4, with 14 cycles of each length.

A cycle of height 3, length ℓ, contains one vertex of height 2 and $\ell - 1$ vertices of height 4. However these vertices are counted twice when both H_t and H_b are equal to 4. From the analysis of the diagram $\Gamma(\mathcal{D})$ in the previous subsection, there are 18 vertices V with

$$H_t(V) = H_b(V) = 4$$

which correspond to the 9 edges of $\Gamma(\mathcal{D})$. For the remaining vertices of height 4, there are 24 with $H_t(V) = 4, H_b(V) = 6$, and 24 with $H_t(V) = 6, H_b(V) = 4$. The total number of vertices of height 4 is therefore equal to 66.

Consider the pure cycles of top type, height 5. Each contains at least one vertex with $H_t(V) = 6, H_b(V) = 4$. But some of these cycles may contain several such vertices. Actually, 13 of these cycles have length 1, hence are not concerned by this problem. There are actually 6 cycles of top type, height 5, length > 1.

- Two have length 2, containing two vertices of height 4.
- One has length 2, containing one vertex of height 4 and one inessential vertex of height 6.
- One has length 2, containing one vertex of height 4 and the essential vertex F of height 6.
- One has length 3, containing three vertices of height 4.
- One has length 3, containing two vertices of height 4 and one inessential vertex of height 6.

There are 5 vertices of height 6: the vertex F has $H_t(F) = H_b(F) = 6$. The other 4 vertices of height 6 are inessential. Two have $H_t(F) = 6, H_b(F) = 8$ and the other two have $H_t(F) = 8, H_b(F) = 6$.

Finally, there are 4 pure cycles of height 7, two of each type. All have length 1.

[21] This pair should be hanging too.

7. The diagram $[6,2](2)(0,0)$

We choose $\mathcal{A}_6$ for alphabet. There is one non trivial automorphism σ of $\mathcal{D}$, associated to the transposition $1 \leftrightarrow -1$. There are two choices of top/bottom exchanging involutions: $\mathcal{I}_0$ is induced by the involution $(-5,5)(-3,3)(-1,1)$ while $\mathcal{I}_1$ is induced by $(-5,5)(-3,3)$.

There are 12 standard vertices. Two of them are fixed by $\mathcal{I}_0$ and exchanged by $\mathcal{I}_1$

$$A_0 := \begin{pmatrix} -5 & -3 & 1 & 3 & -1 & 5 \\ 5 & 3 & -1 & -3 & 1 & -5 \end{pmatrix},\ A_1 := \begin{pmatrix} -5 & -3 & -1 & 3 & 1 & 5 \\ 5 & 3 & 1 & -3 & -1 & -5 \end{pmatrix}.$$

These vertices have default 2.

Another two are fixed by $\mathcal{I}_1$ and exchanged by $\mathcal{I}_0$.

$$D_0 := \begin{pmatrix} -5 & -1 & 1 & -3 & 3 & 5 \\ 5 & -1 & 1 & 3 & -3 & -5 \end{pmatrix},\ D_1 := \begin{pmatrix} -5 & 1 & -1 & -3 & 3 & 5 \\ 5 & 1 & -1 & 3 & -3 & -5 \end{pmatrix}.$$

These vertices have default 5.

Another four vertices have default 3:

$$B_0^+ := \begin{pmatrix} -5 & -3 & 3 & 1 & -1 & 5 \\ 5 & 3 & 1 & -1 & -3 & -5 \end{pmatrix},\quad B_0^- := \begin{pmatrix} -5 & -3 & -1 & 1 & 3 & 5 \\ 5 & 3 & -3 & -1 & 1 & -5 \end{pmatrix},$$

$$B_1^+ := \begin{pmatrix} -5 & -3 & 3 & -1 & 1 & 5 \\ 5 & 3 & -1 & 1 & -3 & -5 \end{pmatrix},\quad B_1^- := \begin{pmatrix} -5 & -3 & 1 & -1 & 3 & 5 \\ 5 & 3 & -3 & 1 & -1 & -5 \end{pmatrix}.$$

The last four standard vertices have default 4:

$$C_0^+ := \begin{pmatrix} -5 & 1 & -3 & -1 & 3 & 5 \\ 5 & 1 & 3 & -3 & -1 & -5 \end{pmatrix},\quad C_0^- := \begin{pmatrix} -5 & -1 & -3 & 3 & 1 & 5 \\ 5 & -1 & 3 & 1 & -3 & -5 \end{pmatrix},$$

$$C_1^+ := \begin{pmatrix} -5 & -1 & -3 & 1 & 3 & 5 \\ 5 & -1 & 3 & -3 & 1 & -5 \end{pmatrix},\quad C_1^- := \begin{pmatrix} -5 & 1 & -3 & 3 & -1 & 5 \\ 5 & 1 & 3 & -1 & -3 & -5 \end{pmatrix}.$$

The non trivial automorphism exchanges[22] B_0^+ and B_1^+, B_0^- and B_1^-, C_0^+ and C_1^+, C_0^- and C_1^-. The involution $\mathcal{I}_0$ exchanges B_0^+ and B_0^-, B_1^+ and B_1^-, C_0^+ and C_0^-, C_1^+ and C_1^-.

Of the 12 vertices linked to any standard vertex, 2 are inessential. The edges of $\Gamma(\mathcal{D})$ are as follows:

- $(A_0 \leftrightarrow C_1^+), (A_0 \leftrightarrow C_1^-), (A_1 \leftrightarrow C_0^+), (A_1 \leftrightarrow C_0^-)$;
- $(B_0^+ \leftrightarrow C_0^-), (B_0^+ \leftrightarrow C_1^-), (B_0^+ \leftrightarrow D_1), (B_0^- \leftrightarrow C_0^+), (B_0^- \leftrightarrow C_1^+), (B_0^- \leftrightarrow D_0)$;
- $(B_1^+ \leftrightarrow C_0^-), (B_1^+ \leftrightarrow C_1^-), (B_1^+, D_0), (B_1^- \leftrightarrow C_0^+), (B_1^- \leftrightarrow C_1^+), (B_1^- \leftrightarrow D_1)$;
- $(C_0^+ \leftrightarrow D_0), (C_0^- \leftrightarrow D_1), (C_1^+ \leftrightarrow D_1), (C_1^- \leftrightarrow D_0)$;
- $(D_0 \leftrightarrow D_1)$.

There are 18 linked open vertices of each type. There are no free vertices.

There are 8 pairs of symmetric deep cycles, all rooted. Their vertices are all linked. Of the deep cycles of top type

- 6 have length 2 with vertices linked to $(A_0, B_1^+), (A_0, B_0^-), (A_1, B_0^+), (A_1, B_1^-)$, $(C_0^+, C_1^-), (C_0^-, C_1^+)$;
- 2 have length 3 with vertices linked to $(A_0, B_0^+, B_1^-), (A_1, B_1^+, B_0^-)$.

[22] D_0 and D_1 are exchanged too.

The default of the diagram is equal to 21. The total number of vertices is

$$N(\mathcal{D}) = 12 \times [(d-1)(d-2)+1] - 42 = 210.$$

It is probably useful to notice that $210 = 6 \times 35$, where 35 was the number of vertices for $[5,2](2)(0)$.

8. THE DIAGRAM $[6,2](0),(2,0)$

The automorphism group of this diagram is cyclic of order 3. Instead of a canonical involution, there are three of them. We choose as alphabet $\{-1 = {}_t\alpha, 1 = {}_b\alpha, 0, a, b, c\}$. The automorphisms fix $-1, 0, 1$ and permute cyclically a, b, c.

There are also 3 top/bottom exchanging involutions I_a, I_b, I_c. The involution I_a exchanges -1 and 1, b and c, and fixes $0, a$.

The diagram has 12 standard vertices.

The involution I_a fixes two standard vertices:

$$P_a := \begin{pmatrix} -1 & 0 & b & c & a & 1 \\ 1 & 0 & c & b & a & -1 \end{pmatrix},\ Q_a := \begin{pmatrix} -1 & b & c & a & 0 & 1 \\ 1 & c & b & a & 0 & -1 \end{pmatrix},$$

and similarly for I_b, I_c. The vertices P_a, P_b, P_c are permuted cyclically by the automorphism group, as are Q_a, Q_b, Q_c. The involution I_a exchanges P_b and P_c, Q_b and Q_c.

The remaining standard vertices are $S_a^+, S_b^+, S_c^+, S_a^-, S_b^-, S_c^-$. One has

$$S_a^+ := \begin{pmatrix} -1 & b & 0 & c & a & 1 \\ 1 & c & b & 0 & a & -1 \end{pmatrix}.$$

The automorphism group permutes cyclically S_a^+, S_b^+, S_c^+ and S_a^-, S_b^-, S_c^-. The involution I_a exchanges S_a^+ and S_a^-, S_b^+ and S_c^-, S_c^+ and S_b^-.

- Of the 12 vertices linked to P_a, 10 are constrained and 2 are inessential. One has $\delta(P_a) = 5$.
- Of the 12 vertices linked to Q_a, 10 are constrained and 2 are inessential. One has $\delta(Q_a) = 5$.
- Of the 12 vertices linked to S_a^+, 8 are constrained, 2 are inessential and 2 are open. One has $\delta(S_a^+) = 4$.

The default of the diagram is $\delta(\mathcal{D}) = 27$.

The edges of $\Gamma(\mathcal{D})$ are as follows

- $(P_a \leftrightarrow Q_a), (P_a \leftrightarrow S_a^+), (P_a \leftrightarrow S_a^-), (P_a \leftrightarrow S_b^-), (P_a \leftrightarrow S_c^+)$;
- $(P_b \leftrightarrow Q_b), (P_b \leftrightarrow S_b^+), (P_b \leftrightarrow S_b^-), (P_b \leftrightarrow S_c^-), (P_b \leftrightarrow S_a^+)$;
- $(P_c \leftrightarrow Q_c), (P_c \leftrightarrow S_c^+), (P_c \leftrightarrow S_c^-), (P_c \leftrightarrow S_a^-), (P_c \leftrightarrow S_b^+)$;
- $(Q_a \leftrightarrow Q_b), (Q_b \leftrightarrow Q_c), (Q_c \leftrightarrow Q_a)$;
- $(Q_a \leftrightarrow S_b^-), (Q_b \leftrightarrow S_c^-), (Q_c \leftrightarrow S_a^-), (Q_a \leftrightarrow S_c^+), (Q_b \leftrightarrow S_a^+), (Q_c \leftrightarrow S_b^+)$;
- $(S_a^+ \leftrightarrow S_c^-), (S_b^+ \leftrightarrow S_a^-), (S_c^+ \leftrightarrow S_b^-)$.

There are no free vertices. There are 6 deep cycles, all of length 2. Their vertices are linked to (S_a^+, S_a^-), (S_b^+, S_b^-), (S_c^+, S_c^-) (twice each).

The total number of vertices is equal to

$$N(\mathcal{D}) = 12 \times [(d-1)(d-2)+1] - 54 = 198.$$

Again, one should notice that $198 = 6 \times 33$.

9. The diagram $[6,2](0),(1,1)$

The automorphism group[23] is cyclic of order 4. We choose for alphabet $\{-1 = {}_t\alpha, 1 = {}_b\alpha, a, b, c, d\}$. There are two top/bottom exchanging involutions I_0, I_1. The first exchanges -1 and 1, a and c, and fixes b, d. The second exchanges -1 and 1, b and d, and fixes a, c. The generator σ of the automorphism group permutes cyclically a, b, c, d in this order.

There are 4 standard vertices S_a, S_b, S_c, S_d permuted cyclically by the automorphism group. One has

$$S_a := \begin{pmatrix} -1 & b & c & d & a & 1 \\ 1 & d & c & b & a & -1 \end{pmatrix}.$$

The involution I_0 fixes S_b and S_d, exchanges S_a and S_c. The involution I_1 fixes S_a and S_c, exchanges S_b and S_d.

Of the 12 vertices linked to S_a, 6 are constrained, 4 are inessential and 2 are open. Similarly for S_b, S_c, S_d. The default of every standard vertex is equal to 3. The default $\delta(\mathcal{D})$ of the diagram is equal to 6.

The graph $\Gamma(\mathcal{D})$ is the full graph on 4 vertices.

There are 8 free vertices, all inessential. There are also 8 deep cycles, each of length 2, consisting of one of the open linked vertices and one of the free vertices.

The total number of vertices is equal to

$$N(\mathcal{D}) = 4 \times [(d-1)(d-2)+1] - 12 + 8 = 80.$$

10. The diagram $[6,2](1),(0,1)$

There are no non trivial automorphisms. We choose for alphabet $\{-2 = {}_t\alpha, 2 = {}_b\alpha, -1, 1, a, b\}$. The involution fixes a and b and exchanges $\pm 1, \pm 2$. There are 4 standard vertices, denoted by P, Q, S^+, S^-. The involution fixes P, Q, exchanges S^+ and S^-.

One has

$$P := \begin{pmatrix} -2 & a & -1 & b & 1 & 2 \\ 2 & a & 1 & b & -1 & -2 \end{pmatrix}, \ Q := \begin{pmatrix} -2 & -1 & b & a & 1 & 2 \\ 2 & 1 & b & a & -1 & -2 \end{pmatrix},$$
$$S^+ := \begin{pmatrix} -2 & -1 & a & b & 1 & 2 \\ 2 & 1 & b & -1 & a & -2 \end{pmatrix}, \ S^- := \begin{pmatrix} -2 & -1 & b & 1 & a & 2 \\ 2 & 1 & a & b & -1 & -2 \end{pmatrix}.$$

- Of the 12 vertices linked to P, 6 are constrained, 4 are inessential and 2 are open. One has $\delta(P) = 3$;
- Of the 12 vertices linked to Q, 2 are constrained, 4 are inessential and 6 are open. One has $\delta(P) = 1$;
- Of the 12 vertices linked to $S^{\pm}$, 2 are constrained, 4 are inessential and 6 are open. One has $\delta(S^{\pm}) = 1$.

The default $\delta(\mathcal{D})$ of the diagram is equal to 3. In $\Gamma(\mathcal{D})$, the only edges are the ones linking P to every other vertex.

There are 12 free vertices, 8 of them inessential and 16 deep cycles.

- Each of the two open vertices linked to P belongs to a deep cycle of length 2, whose other vertex is free and inessential.

---

[23]From [Boissy] (see footnote 7) we see that the group has order 4. One can obtain a generator σ by considering the monotonous chain of length 4 corresponding to the pair (a, b).

- There are two deep cycles of length 2 (one of each type), whose vertices are open and linked to (Q, S^+); similarly ,there are two deep cycles of length 2, whose vertices are open and linked to (Q, S^-);
- There are two deep cycles of length 3, one of each type, containing one vertex linked to Q, one vertex which is free but inessential, and one free vertex $F^{\pm}$;
- There are two other deep cycles of length 3, one of each type, containing one vertex linked to S^+, one vertex linked to S^- and one free inessential vertex;
- There is one deep cycle of length 2 containing a vertex linked to S^+ and a free inessential vertex; similarly for S^-;
- Finally, there are two other symmetric free essential vertices $G^{\pm}$. Both deep cycles through G^+ have length 2, the other vertex being F^+ (for one cycle) and a vertex linked to S^+ (for the other).

The total number of vertices is equal to

$$N(\mathcal{D}) = 4 \times [(d-1)(d-2)+1] - 6 + 12 = 90.$$

Perhaps one should observe that $90 = 6 \times 15$, where 15 is the number of vertices of the hyperelliptic diagram for $d = 5$.

11. The diagram $[7,3](3)(1)$

11.1. **Alphabet, Automorphisms, Involution.** We take as alphabet $\mathcal{A} = \{\pm\infty, \pm 1 \pm 2, 0\}$. There is no nontrivial automorphism. The involution exchanges $\pm\infty$, ± 1, ± 2 and fixes 0.

11.2. **Standard vertices.** There are 16 standard vertices. Two of them are fixed by the involution

$$S := \begin{pmatrix} -\infty & 2 & 0 & -2 & 1 & -1 & +\infty \\ +\infty & -2 & 0 & 2 & -1 & 1 & -\infty \end{pmatrix},$$

$$T := \begin{pmatrix} -\infty & -2 & -1 & 1 & 0 & 2 & +\infty \\ +\infty & 2 & 1 & -1 & 0 & -2 & -\infty \end{pmatrix}.$$

Otherwise, we have 7 pairs of symmetric vertices

$$A^+ := \begin{pmatrix} -\infty & 1 & 0 & -2 & -1 & 2 & +\infty \\ +\infty & 0 & 2 & 1 & -1 & -2 & -\infty \end{pmatrix},$$

$$A^- := \begin{pmatrix} -\infty & 0 & -2 & -1 & 1 & 2 & +\infty \\ +\infty & -1 & 0 & 2 & 1 & -2 & -\infty \end{pmatrix},$$

$$B^+ := \begin{pmatrix} -\infty & 0 & -2 & 1 & -1 & 2 & +\infty \\ +\infty & -1 & -2 & 0 & 2 & 1 & -\infty \end{pmatrix},$$

$$B^- := \begin{pmatrix} -\infty & 1 & 2 & 0 & -2 & -1 & +\infty \\ +\infty & 0 & 2 & -1 & 1 & -2 & -\infty \end{pmatrix},$$

$$C^+ := \begin{pmatrix} -\infty & -2 & 1 & -1 & 2 & 0 & +\infty \\ +\infty & 2 & -1 & -2 & 0 & 1 & -\infty \end{pmatrix},$$

$$C^- := \begin{pmatrix} -\infty & -2 & 1 & 2 & 0 & -1 & +\infty \\ +\infty & 2 & -1 & 1 & -2 & 0 & -\infty \end{pmatrix},$$

$$D^+ := \begin{pmatrix} -\infty & 1 & -2 & -1 & 2 & 0 & +\infty \\ +\infty & 0 & 1 & 2 & -1 & -2 & -\infty \end{pmatrix},$$

$$D^- := \begin{pmatrix} -\infty & 0 & -1 & -2 & 1 & 2 & +\infty \\ +\infty & -1 & 2 & 1 & -2 & 0 & -\infty \end{pmatrix},$$

$$E^+ := \begin{pmatrix} -\infty & -2 & -1 & 1 & 2 & 0 & +\infty \\ +\infty & 2 & -1 & 0 & 1 & -2 & -\infty \end{pmatrix},$$

$$E^- := \begin{pmatrix} -\infty & -2 & 1 & 0 & -1 & 2 & +\infty \\ +\infty & 2 & 1 & -1 & -2 & 0 & -\infty \end{pmatrix},$$

$$F^+ := \begin{pmatrix} -\infty & 1 & -1 & 2 & 0 & -2 & +\infty \\ +\infty & 0 & 2 & -1 & -2 & 1 & -\infty \end{pmatrix},$$

$$F^- := \begin{pmatrix} -\infty & 0 & -2 & 1 & 2 & -1 & +\infty \\ +\infty & -1 & 1 & -2 & 0 & 2 & -\infty \end{pmatrix},$$

$$G^+ := \begin{pmatrix} -\infty & -2 & -1 & 2 & 1 & 0 & +\infty \\ +\infty & 2 & 0 & 1 & -1 & -2 & -\infty \end{pmatrix},$$

$$G^- := \begin{pmatrix} -\infty & -2 & 0 & -1 & 1 & 2 & +\infty \\ +\infty & 2 & 1 & -2 & -1 & 0 & -\infty \end{pmatrix}.$$

11.3. **The diagram $\Gamma(\mathcal{D})$.** The vertex S has default 6, with edges to $C^+, C^-, B^+, B^-, F^+, F^-$.
The vertex T has default 2, with edges to A^+, A^-.
The vertex A^+ has default 5, with edges to G^+, A^-, B^+, E^-, T.
The vertex B^+ has default 5, with edges to S, A^+, F^+, E^-, C^+.
The vertex C^+ has default 4, with edges to D^+, S, B^+, F^+.
The vertex D^+ has default 3, with edges to C^+, G^+, E^+.
The vertex E^+ has default 3, with edges to A^-, B^-, D^+.
The vertex F^+ has default 3, with edges to S, B^+, C^+.
The vertex G^+ has default 2, with edges to A^+, D^+.
The default of the diagram is $\delta(\mathcal{D}) = 29$.

The model for a monotonous chain of length 7 connects G^+ and G^-. In $\Gamma(\mathcal{D})$, the shortest way is to use the edges from G^+ to A^+, A^+ to A^-, A^- to G^-.

11.4. **Vertices of height $\leqslant 4$.** There are 16 pure cycles of each type, height 1, each with 5 vertices of height 2. This gives altogether 160 vertices of height 2. Attached to these vertices are 160 pure cycles of height 3. Actually, for each $1 \leqslant \ell \leqslant 5$, there are 16 cycles of each type, height 3 and length ℓ. Such a cycle contains $\ell - 1$ vertices of height 4.

There are $160 = 16 \times 10$ vertices with $H_t(\pi) = H(\pi) = 4$, and similarly $160 = 16 \times 10$ vertices with $H_b(\pi) = H(\pi) = 4$. In view of the default of $\Gamma(\mathcal{D})$, this gives 58 vertices with $H_t(\pi) = H_b(\pi) = 4$, 102 vertices with $H_t(\pi) = 4, H_b(\pi) = 6$, and 102 vertices with $H_t(\pi) = 6, H_b(\pi) = 4$.

Let V be a vertex with $H_t(V) = 6, H_b(V) = 4$. Let $(V_0, C_1, V_2, C_3, V_4 = V)$ be the chain connecting V to a standard vertex V_0. Let α_t, α_b be the winners of the top and bottom cycles through V. In V_0, we have

$$\pi_t(\alpha_b) < \pi_t(\alpha_t), \quad \pi_b(\alpha_t) < \pi_b(\alpha_b).$$

Moreover, the length of the cycle C_5 of top type through V is equal to $\pi_b(\alpha_b) - \pi_b(\alpha_t)$. The vertices $V' \neq V$ in C_5 with $H_t(V) = 6, H_b(V) = 4$ (i.e $H(V') = 4$ as $H_t(V') = 6$

is automatic) correspond to the letters α'_b such that $\pi_b(\alpha_t) < \pi_b(\alpha'_b) < \pi_b(\alpha_b)$ and $\alpha'_b \notin [\alpha_b \nearrow \alpha_t]_t$.

We consider the pure cycles of height 5 linked to the different standard vertices

- There are 4 cycles of each type linked to S; 3 have length 1 and one has length 2, associated to $(\alpha_t, \alpha_b) = (2, -2)$.
- There are 8 cycles of each type linked to T; 3 (of each type) have length 1, 2 have length 2, 2 have length 3 and one has length 4.
- There are 5 cycles of each type linked to A^+. Amongst these, 5 have length 1, 3 have length 2 (2 top, 1 bottom), 1 (top) has length 3 and 1 (bottom) has length 4.
- There are 5 cycles of each type linked to B^+. Amongst these, 5 have length 1, 3 have length 2 (1 top, 2 bottom), 1 (bottom) has length 3 and 1 (top) has length 4.
- There are 6 cycles of each type linked to C^+. Amongst these, 5 have length 1, 3 have length 2 (1 top, 2 bottom), 3 (1 top, 2 bottom) has length 3 and 1 (top) has length 4.
- There are 7 cycles of each type linked to D^+. Amongst these, 6 have length 1, 4 have length 2 (2 top, 2 bottom), 2 (1 top, 1 bottom) has length 3 and 2 (1 top, 1 bottom) has length 4.
- There are 7 cycles of each type linked to E^+. Amongst these, 5 have length 1, 4 have length 2 (1 top, 3 bottom), 3 (2 top, 1 bottom) has length 3 and 2 (1 top, 1 bottom) has length 4.
- There are 7 cycles of each type linked to F^+. Amongst these, 6 have length 1, 4 have length 2 (2 top, 2 bottom), 2 (1 top, 1 bottom) has length 3 and 2 (1 top, 1 bottom) has length 4.
- There are 8 cycles of each type linked to G^+. Amongst these, 6 have length 1, 4 have length 2 (2 top, 2 bottom), 4 (2 top, 2 bottom) has length 3 and 2 (1 top, 1 bottom) has length 4.

Summarizing, there are 44 cycles of top type, height 5 and length 1. For length > 1 we need to know how many times each cycle is counted, i.e how many vertices of height 4 these cycles contain.

11.5. **Cycles of height 5 and vertices of height 6.** There are 22 pure cycles of top type, height 5 and length 2. Among these, 16 contain a vertex of height 4 and a vertex of height 6, and 6 contain two vertices of height 4. These 6 cycles are the midcycles of monotonous chains of length 5 connecting T to E^-, A^+ to D^+, B^- to A^+, C^- to E^-, E^- to G^- and F^+ to B^-.

There are 10 pure cycles of top type, height 5 and length 3. Two of these cycles have only vertices of height 4. The first one connects T, E^+, G^+, the second one C^+, C^-, E^+. Four of these cycles have one vertex of height 6 and two of height 4 (connecting (T, G^-), (A^+, F^+), (B^-, D^+), (C^-, G^-) respectively. Finally four of these cycles have two vertices of height 6 and one of height 4 (connected to D^-, E^-, F^-, G^+ respectively).

There are 5 pure cycles of top type, height 5 and length 4. One of these cycles have one vertex of height 4 (connected to T) and three of height 6. Two of these cycles have two vertices of height 4 (connected to (D^+, F^+), (E^+, G^-) respectively) and two of height 6. One of these cycles have three vertices of height 4 (connected to C^+, G^+, E^-) and one vertex of height 6. The last cycle has four vertices of height 4, connected to A^-, D^-, F^-, B^+.

We conclude that altogether there are $36 = 16+12+8$ vertices with $H_t(V) = H(V) = 6$. Some of these vertices will have $H_b(V) = 6$, the others will have $H_b(V) = 8$.

For these vertices V with $H_t(V) = H(V) = 6$, denote by C the pure cycle of bottom type through V. For 21 of these vertices, C has length 1 (hence height 7). For 7 of these vertices , C has height 5: these cases correspond to monotonous chains of length 6 connecting T to B^-, B^+ to T, F^- to F^+, C^- to F^+, F^- to C^+, F^- to E^+, E^- to F^+. For the last 8 vertices, C has length > 1 and height 7. More precisely

- There is a monotonous chain of length 7 between T and E^+;
- There is a monotonous chain of length 7 between G^+ and G^-;
- In two cases (connected to D^+, D^-, C has height 7, length 2. The other vertex in C has height 8, and the top cycle through this other vertex has length 1 (hence height 9);
- The last case is a monotonous chain of length 7 from T to G^+ with a decoration: C has length 3, there is an additional vertex of height 8 such that the top cycle through it has length 1 (hence height 9).

We have only described the cases where C is of bottom type. The other cases are obtained from the involution.

There are 7 vertices with $H_t(V) = H_b(V) = 6$, 29 with $H_t(V) = 6, H_b(V) = 8$, 29 with $H_t(V) = 8, H_b(V) = 6$. There are 5 pure cycles of each type of height 7 and length > 1, 4 of length 2 and one of length 3. Finally, there are 3 vertices with $H_t(V) = 8, H_b(V) = 10$, and 3 with $H_t(V) = 10, H_b(V) = 8$.

Summarizing, there are

- 16 vertices of height 0;
- 160 vertices of height 2;
- 262 vertices of height 4;
- 65 vertices of height 6;
- 6 vertices of height 8;

Apparently, the diagram has 509 vertices.

12. THE DIAGRAMS $[4+N,2](2)(0^N)$

We have already seen the cases $N = 0, 1, 2$ from which we infer the general case.

12.1. Alphabet, automorphism group and involution. The alphabet is $\mathcal{A} = \mathcal{A}_4 \sqcup \mathcal{A}^*$, where $\mathcal{A}^*$ is an alphabet on N letters. The automorphism group is the group of permutations of $\mathcal{A}^*$. The involution exchanges 3 and -3, 1 and -1, and fixes every letter in $\mathcal{A}^*$.

12.2. Standard vertices. Standard vertices are in one-to-one correspondence with triples (a, b, c) where

- a (resp.b, resp. c) is a bijection from $\{1, \dots, |a|\}$ (resp. $\{1, \dots, |b|\}$, resp. $\{1, \dots, |c|\}$) onto a subset A (resp. B, resp. C) of $\mathcal{A}^*$;
- the subsets A, B, C form a partition of $\mathcal{A}^*$.

The subsets A, B, C are allowed to be empty (this corresponds to $|a| = 0, \dots$).

The standard vertex associated to (a, b, c) is

$$S(a,b,c) := \begin{pmatrix} -3 & a & -1 & b & 1 & c & 3 \\ 3 & a & 1 & c & -1 & b & -3 \end{pmatrix}.$$

Here, a in the top or bottom line means $(a(1), \dots, a(|a|))$.

The number of standard vertices is equal to

$$N_{st}(\mathcal{D}) = \frac{1}{2}(N+2)!\ .$$

12.3. **Default of standard vertices.** For $S(a,b,c)$ as above, let us compute the number of pairs of distinct letters in $\mathcal{A}$ which are ordered in the same way by π_t and π_b. Notice that at least one of these letters must belong to $\mathcal{A}^*$. If (α,β) is such a pair, we have the following possibilities

(1) Both α, β belong to a;
(2) Both α, β belong to $b \cup \{-1\}$;
(3) Both α, β belong to $c \cup \{1\}$;
(4) α belongs to a, and β belongs to $b \cup \{-1\}$;
(5) α belongs to a, and β belongs to $c \cup \{1\}$;

The default of $S(a,b,c)$ is thus equal to

$$\begin{aligned}\delta(S(a,b,c)) &= N(|a|+1) + \frac{|b|(|b|-1)}{2} + \frac{|c|(|c|-1)}{2} - \frac{|a|(|a|-1)}{2} \\ &= \frac{N(N+1)}{2} - |b||c| + |a|.\end{aligned}$$

The minimum value is $\lfloor \frac{(N+1)^2}{4} \rfloor$ (when $|a| = 0$ and $||b| - |c|| \leqslant 1$). The maximum value is $\frac{N(N+3)}{2}$ (when $|a| = N$).

Of the $(N+2)(N+1)$ vertices linked to any standard vertex, only 2 are inessential, one on each side of the standard vertex. On the top side of the standard vertex, the inessential linked vertex has $\alpha_b = -1$.

12.4. **Edges of $\Gamma(\mathcal{D})$.** To each pair (α,β) as in the last subsection corresponds an edge of $\Gamma(\mathcal{D})$ from $S(a,b,c)$ to another standard vertex $S(a',b',c')$.

(1) $\alpha, \beta \in a$: We write $a = a_0a_1a_2$, with a_0, a_1 non empty. We have

$$a' = a_1a_0a_2, \quad b' = b, \quad c' = c.$$

(2) $\alpha, \beta \in b \cup \{-1\}$: We write $b = b_0b_1b_2$, with b_1 non empty. We have

$$a' = b_1a, \quad b' = b_0b_2, \quad c' = c.$$

The case where b_0 is empty corresponds to $\alpha = -1$.

(3) $\alpha, \beta \in c \cup \{1\}$: We write $c = c_0c_1c_2$, with c_1 non empty. We have

$$a' = c_1a, \quad b' = b, \quad c' = c_0c_2.$$

The case where c_0 is empty corresponds to $\alpha = 1$.

(4) $\alpha \in a, \beta \in b \cup \{-1\}$: We write $a = a_0a_1, b = b_0b_1$ with a_0 non empty. We have

$$a' = a_1, \quad b' = b_0a_0b_1, \quad c' = c.$$

(5) $\alpha \in a, \beta \in c \cup \{1\}$: We write $a = a_0a_1, c = c_0c_1$ with a_0 non empty. We have

$$a' = a_1, \quad b' = b, \quad c' = c_0a_0c_1.$$

Notice that the edge (1) leaves $|a|, b, c$ unchanged. The edge (2) (resp. (3)) lengthens a and shortens b (resp. c); it is the opposite of (4) (resp. (5)).

12.5. **Open linked vertices.** Let $S(a,b,c)$ as above. To get the unconstrained vertices which are linked to it, take $\beta \in b \cup \{-1\}$, $\kappa \in c \cup \{1\}$ and write $b = b_0 b_1$, $c = c_0 c_1$. The two vertices corresponding to (β, κ) are

$$\begin{pmatrix} -3 & a & -1 & b_0 & c_1 & 3 & b_1 & 1 & c_0 \\ 3 & b_1 & -3 & a & 1 & c_0 & c_1 & -1 & b_0 \end{pmatrix},$$

$$\begin{pmatrix} -3 & c_1 & 3 & a & -1 & b_0 & b_1 & 1 & c_0 \\ 3 & a & 1 & c_0 & b_1 & -3 & c_1 & -1 & b_0 \end{pmatrix}.$$

Observe that b_0, b_1, c_0, c_1 may be empty! The first vertex is inessential iff b_0 and c_1 are empty. The second vertex is inessential iff c_0 and b_1 are empty.

The first vertex belongs to a deep cycle of top type of length $1 + |b_0| + |c_1|$. All other vertices in this cycle are also linked to some standard vertex, hence there is no free vertex! These other vertices are associated to decompositions $b_0 = b_0^{(1)} b_0^{(2)}$ with $b_0^{(2)}$ non empty **or** to decompositions $c_1 = c_1^{(1)} c_1^{(2)}$ with $c_1^{(1)}$ non empty.

The vertex associated to $b_0 = b_0^{(1)} b_0^{(2)}$ is linked to $S(a', b', c')$ with

$$a' = a, \quad b' = b_0^{(1)} b_1, \quad c' = c_0 b_0^{(2)} c_1.$$

The vertex associated to $c_1 = c_1^{(1)} c_1^{(2)}$ is linked to $S(a', b', c')$ with

$$a' = a, \quad b' = b_0 c_1^{(1)} b_1, \quad c' = c_0 c_1^{(2)}.$$

Observe that these cycles keep a fixed.

12.6. **The total number of vertices.** One first compute the default of the diagram. One obtains, after a small computation[24]

$$\delta(\mathcal{D}) = \frac{1}{2} \sum \delta(S) = N! \frac{N(N+1)(N+2)(5N+11)}{48}.$$

The final result is

$$N(\mathcal{D}) = N_{st}(\mathcal{D})(N^2 + 5N + 7) - 2\delta(\mathcal{D}) = \frac{7}{24}(N+4)!$$

13. The diagrams $[4+N, 2](0)(2, 0^{N-1})$

13.1. **Alphabet, Automorphisms, Involution.** The alphabet is $\mathcal{A} = \{-1, 1, a, b, c\} \sqcup \mathcal{A}^*$, where $\mathcal{A}^*$ has $N-1$ letters. The automorphism group is the product of the symmetric group of $\mathcal{A}^*$ and a cyclic group of order 3 which permutes cyclically a, b, c. There are three involutions I_a, I_b, I_c. The involution I_a fixes a and every letter in $\mathcal{A}^*$, exchanges -1 and 1, and also b and c. Similarly for I_b, I_c. The letters a, b, c are associated to the three pairs of vertical separatrices of the double zero[25], the letters in $\mathcal{A}^*$ to the $N-1$ nonsingular marked points which are not the root of the RV algorithm.

[24]See Section 16.5 for some formulas that are used there and in the next sections.

[25]The relation between the letters and the separatrices (horizontal, or vertical) can be found in the paper of C. Boissy mentionned in footnote 7.

13.2. **Standard vertices.** The standard vertices are parametrized by a letter $x \in \{a,b,c\}$ and a 4-tuple $w = (w_0, w_a, w_b, w_c)$ where w_i, for $i \in \{0,a,b,c\}$, is a bijection of $\{1,\dots,|w_i|\}$ onto a subset A_i of $\mathcal{A}^*$, and the A_i form a partition of $\mathcal{A}^*$. We have

$$S_c(w) := \begin{pmatrix} -1 & w_0 & a & w_a & b & w_b & c & w_c & 1 \\ 1 & w_0 & b & w_b & a & w_a & c & w_c & -1 \end{pmatrix}.$$

The number of standard vertices is equal to

$$N_{st}(\mathcal{D}) = \frac{1}{2}(N+2)!.$$

13.3. **Edges of $\Gamma(\mathcal{D})$.** The pairs of letters (α,β) which are ordered in $S_c(w)$ in the same way by π_t and π_b are divided in several types. We denote by $x \in \{a,b,c\}$ and $w' = (w'_0, w'_a, w'_b, w'_c)$ the symbols such that $S_c(w)$ is connected via the (α,β) edge to $S_x(w')$.

(1) $\alpha,\beta \in w_0$: this gives, with $w_0 = w_0^{(1)} w_0^{(2)} w_0^{(3)}$

$$x = c, \quad w'_0 = w_0^{(2)} w_0^{(1)} w_0^{(3)}, \quad w'_a = w_a, \quad w'_b = w_b, \quad w'_c = w_c.$$

(2) $\alpha,\beta \in \{a\} \cup w_a$: this gives, with $w_a = w_a^{(1)} w_a^{(2)} w_a^{(3)}$

$$x = c, \quad w'_0 = w_a^{(2)} w_0, \quad w'_a = w_a^{(1)} w_a^{(3)}, \quad w'_b = w_b, \quad w'_c = w_c.$$

(3) $\alpha,\beta \in \{b\} \cup w_b$: this gives, with $w_b = w_b^{(1)} w_b^{(2)} w_b^{(3)}$

$$x = c, \quad w'_0 = w_b^{(2)} w_0, \quad w'_a = w_a, \quad w'_b = w_b^{(1)} w_b^{(3)}, \quad w'_c = w_c.$$

(4) $\alpha,\beta \in \{c\} \cup w_c$: this gives, with $w_c = w_c^{(1)} w_c^{(2)} w_c^{(3)}$

$$x = c, \quad w'_0 = w_c(2) w_0, \quad w'_a = w_a, \quad w'_b = w_b, \quad w'_c = w_c^{(1)} w_c^{(3)}.$$

(5) $\alpha \in w_0, \beta \in \{a\} \cup w_a$: we write $w_0 = w_0^{(1)} w_0^{(2)}$ and $w_a = w_a^{(1)} w_a^{(2)}$ to obtain

$$x = c, \quad w'_0 = w_0^{(2)}, \quad w'_a = w_a^{(1)} w_0^{(1)} w_a^{(2)}, \quad w'_b = w_b, \quad w'_c = w_c.$$

(6) $\alpha \in w_0, \beta \in \{b\} \cup w_b$: we write $w_0 = w_0^{(1)} w_0^{(2)}$ and $w_b = w_b^{(1)} w_b^{(2)}$ to obtain

$$x = c, \quad w'_0 = w_0^{(2)}, \quad w'_a = w_a, \quad w'_b = w_b^{(1)} w_0^{(1)} w_b^{(2)}, \quad w'_c = w_c.$$

(7) $\alpha \in w_0, \beta \in \{c\} \cup w_c$: we write $w_0 = w_0^{(1)} w_0^{(2)}$ and $w_c = w_c^{(1)} w_c^{(2)}$ to obtain

$$x = c, \quad w'_0 = w_0^{(2)}, \quad w'_a = w_a, \quad w'_b = w_b, \quad w'_c = w_c^{(1)} w_0^{(1)} w_c^{(2)}.$$

(8) $\alpha \in \{a\} \cup w_a, \beta \in \{c\} \cup w_c$: we write $w_a = w_a^{(1)} w_a^{(2)}$ and $w_c = w_c^{(1)} w_c^{(2)}$ to obtain

$$x = a, \quad w'_0 = w_a^{(2)}, \quad w'_b = w_b, \quad w'_c = w_c^{(1)} w_0, \quad w'_a = w_a^{(1)} w_c^{(2)}.$$

(9) $\alpha \in \{b\} \cup w_b, \beta \in \{c\} \cup w_c$: we write $w_b = w_b^{(1)} w_b^{(2)}$ and $w_c = w_c^{(1)} w_c^{(2)}$ to obtain

$$x = b, \quad w'_0 = w_b^{(2)}, \quad w'_c = w_c^{(1)} w_0, \quad w'_a = w_a, \quad w'_b = w_b^{(1)} w_c^{(2)}.$$

A small computation gives the default of the vertex $S_c(w)$.

$$\delta(S_c(w)) = \frac{N(N+1)}{2} - |w_a||w_b| + |w_0| + |w_c| + 1.$$

The default is minimal when w_0, w_c are empty and $||w_a| - |w_b|| \leqslant 1$, maximal when w_a, w_b are empty.

It is easy to see that $\Gamma(\mathcal{D})$ is connected and thus that the standard vertices are as claimed. Indeed take as base point a vertex of $\Gamma(\mathcal{D})$ such that $x = c$ and w_a, w_b, w_c are empty. Starting with any other vertex, an edge of type (8) or (9) (if necessary) leads to a vertex with $x = c$. Then edges of type (2), (3), (4) allow to eliminate w_a, w_b, w_c. Finally, a succession of edges of type (1) connect to the base point.

13.4. **Default of the diagram.** A small computation gives

$$\begin{aligned}\delta(\mathcal{D}) &= 3(N-1)! \sum_{n_0+n_a+n_b+n_c=N-1} [\frac{N(N+1)}{2} + 1 + n_0 + n_c - n_a n_b] \\ &= (N+2)!\, \frac{9N^2+23N+8}{80}.\end{aligned}$$

13.5. **Open linked vertices.** Each pair (α, β) with $\alpha \in \{a\} \cup w_a,\ \beta \in \{b\} w_b$ gives rise to a pair of unconstrained linked vertices. Writing $w_a = w_a^{(1)} w_a^{(2)},\ w_b = w_b^{(1)} w_b^{(2)}$, these vertices are

$$\begin{pmatrix} -1 & w_0 & a & w_a^{(1)} & w_b^{(2)} & c & w_c & 1 & w_a^{(2)} & b & w_b^{(1)} \\ 1 & w_a^{(2)} & c & w_c & -1 & w_0 & b & w_b^{(1)} & w_b^{(2)} & a & w_a^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} -1 & w_b^{(2)} & c & w_c & 1 & w_0 & a & w_a^{(1)} & w_a^{(2)} & b & w_b^{(1)} \\ 1 & w_0 & b & w_b^{(1)} & w_a^{(2)} & c & w_c & -1 & w_b^{(2)} & a & w_a^{(1)} \end{pmatrix}.$$

The first (second) vertex is part of a deep cycle of top (bottom) type of length $|w_b^{(2)}| + |w_a^{(1)}| + 1$ (resp. $|w_a^{(2)}| + |w_b^{(1)}| + 1$). Therefore this vertex is inessential iff $|w_b^{(2)}| = |w_a^{(1)}| = 0$ (resp. $|w_a^{(2)}| = |w_b^{(1)}| = 0$).

All vertices in these deep cycles are linked to some standard vertex, hence there are no free vertices.

13.6. **Number of vertices.** As there are no free vertices the total number of vertices is given by

$$\begin{aligned}N(\mathcal{D}) &= \frac{1}{2}(N+2)!(N^2+5N+7) - 2\delta(\mathcal{D}) \\ &= \frac{11}{40}(N+4)!\end{aligned}$$

14. The diagrams $[5+N, 2](1)(1, 0^N)$

14.1. **Alphabet, Automorphisms, Involution.** The alphabet is $\mathcal{A} = \{-2, -1, 0, 1, 2\} \sqcup \mathcal{A}^*$, with $|\mathcal{A}^*| = N$.

The involution fixes each letter in $\mathcal{A}^*$ and 0, and exchanges ± 1, ± 2.

The automorphism group is the symmetric group of $\mathcal{A}^*$.

14.2. **Standard vertices.** For each $w = (w_-, w_{-1}, w_0, w_1)$, we have the vertex

$$S(w) := \begin{pmatrix} -2 & w_- & -1 & w_{-1} & 0 & w_0 & 1 & w_1 & 2 \\ 2 & w_- & 1 & w_1 & 0 & w_0 & -1 & w_{-1} & -2 \end{pmatrix}.$$

The number of standard vertices is equal to

$$N_{st}(\mathcal{D}) = \frac{1}{6}(N+3)!.$$

14.3. **Edges of $\Gamma(\mathcal{D})$.** The pairs of letters (α, β) which are ordered in $S(w)$ in the same way by π_t and π_b are divided in several types. We denote by $w' = (w'_0, w'_a, w'_b, w'_c)$ the symbols such that $S(w)$ is connected via the (α, β) edge to $S(w')$.

(1) $\alpha, \beta \in w_-$: We write $w_- = w_-^{(1)} w_-^{(2)} w_-^{(3)}$ and we have

$$w'_- = w_-^{(2)} w_-^{(1)} w_-^{(3)}, \quad w'_{-1} = w_{-1}, \quad w'_0 = w_0, \quad w'_1 = w_1.$$

(2) $\alpha, \beta \in \{-1\} \cup w_{-1}$: We write $w_{-1} = w_{-1}^{(1)} w_{-1}^{(2)} w_{-1}^{(3)}$ and we have

$$w'_- = w_{-1}^{(2)} w_-, \quad w'_{-1} = w_{-1}^{(1)} w_{-1}^{(3)}, \quad w'_0 = w_0, \quad w'_1 = w_1.$$

(3) $\alpha, \beta \in \{0\} \cup w_0$: We write $w_0 = w_0^{(1)} w_0^{(2)} w_0^{(3)}$ and we have

$$w'_- = w_0^{(2)} w_-, \quad w'_{-1} = w_{-1}, \quad w'_0 = w_0^{(1)} w_0^{(3)}, \quad w'_1 = w_1.$$

(4) $\alpha, \beta \in \{1\} \cup w_1$: We write $w_1 = w_1^{(1)} w_1^{(2)} w_1^{(3)}$ and we have

$$w'_- = w_1^{(2)} w_-, \quad w'_{-1} = w_{-1}, \quad w'_0 = w_0, \quad w'_1 = w_1^{(1)} w_1^{(3)}.$$

(5) $\alpha \in w_-, \beta \in \{-1\} \cup w_{-1}$: We write $w_- = w_-^{(1)} w_-^{(2)}$, $w_{-1} = w_{-1}^{(1)} w_{-1}^{(2)}$ and we have

$$w'_- = w_-^{(2)}, \quad w'_{-1} = w_{-1}^{(1)} w_-^{(1)} w_{-1}^{(2)}, \quad w'_0 = w_0, \quad w'_1 = w_1.$$

(6) $\alpha \in w_-, \beta \in \{0\} \cup w_0$: We write $w_- = w_-^{(1)} w_-^{(2)}$, $w_0 = w_0^{(1)} w_0^{(2)}$ and we have

$$w'_- = w_-^{(2)}, \quad w'_{-1} = w_{-1}, \quad w'_0 = w_0^{(1)} w_-^{(1)} w_0^{(2)}, \quad w'_1 = w_1.$$

(7) $\alpha \in w_-, \beta \in \{1\} \cup w_1$: We write $w_- = w_-^{(1)} w_-^{(2)}$, $w_1 = w_1^{(1)} w_1^{(2)}$ and we have

$$w'_- = w_-^{(2)}, \quad w'_{-1} = w_{-1}, \quad w'_0 = w_0, \quad w'_1 = w_1^{(1)} w_-^{(1)} w_1^{(2)}.$$

A small computation gives the default of the vertex $S(w)$.

$$\delta(S(w)) = \frac{N(N+1)}{2} + 2|w_-| - (|w_{-1}||w_0| + |w_0||w_1| + |w_1||w_{-1}|).$$

The default is minimal, equal to $\lfloor \frac{(N+1)(N+2)}{6} \rfloor$, when w_- is empty and

$$||w_{-1}| - |w_0|| \leqslant 1, \quad ||w_0| - |w_1|| \leqslant 1, \quad ||w_1| - |w_{-1}|| \leqslant 1.$$

The default is maximal, equal to $\frac{N(N+5)}{2}$, when w_{-1}, w_0, w_1 are empty.

The proof that $\Gamma(\mathcal{D})$ is connected, using the first four types of edges, is as in the last section. This implies that the list of standard vertices is as stated.

14.4. **Default of the diagram.** A small computation gives

$$\begin{aligned}\delta(\mathcal{D}) &= \frac{1}{2}N! \sum_{n_-+n_{-1}+n_0+n_1=N} [\frac{N(N+1)}{2} + 2n_- \ - 3n_0 n_1] \\ &= (N+3)!\ \frac{N(7N+23)}{240}.\end{aligned}$$

14.5. **Open linked vertices.** Unconstrained linked vertices are obtained from three types of pairs (α, β) of letters of $\mathcal{A}$.

(1) $\alpha \in \{-1\} \cup w_{-1}, \beta \in \{0\} \cup w_0$: Write $w_{-1} = w_{-1}^{(1)} w_{-1}^{(2)}$, $w_0 = w_0^{(1)} w_0^{(2)}$. The two vertices obtained from (α, β) are

$$\begin{pmatrix} -2 & w_- & -1 & w_{-1}^{(1)} & w_0^{(2)} & 1 & w_1 & 2 & w_{-1}^{(2)} & 0 & w_0^{(1)} \\ 2 & w_{-1}^{(2)} & -2 & w_- & 1 & w_1 & 0 & w_0^{(1)} & w_0^{(2)} & -1 & w_{-1}^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} -2 & w_0^{(2)} & 1 & w_1 & 2 & w_- & -1 & w_{-1}^{(1)} & w_{-1}^{(2)} & 0 & w_0^{(1)} \\ 2 & w_- & 1 & w_1 & 0 & w_0^{(1)} & w_{-1}^{(2)} & -2 & w_0^{(2)} & -1 & w_{-1}^{(1)} \end{pmatrix}.$$

There is a deep cycle of top type through the first vertex but all the vertices in this cycle are linked to some standard vertex.

Similarly, there is a deep cycle of bottom type through the second vertex but all the vertices in this cycle are linked to some standard vertex.

(2) $\alpha \in \{1\} \cup w_1, \beta \in \{0\} \cup w_0$: this case is similar to the first one.

(3) $\alpha \in \{-1\} \cup w_{-1}, \beta \in \{1\} \cup w_1$: Write $w_{-1} = w_{-1}^{(1)} w_{-1}^{(2)}$, $w_1 = w_1^{(1)} w_1^{(2)}$. The two vertices obtained from (α, β) are

$$A_t := \begin{pmatrix} -2 & w_- & -1 & w_{-1}^{(1)} & w_1^{(2)} & 2 & w_{-1}^{(2)} & 0 & w_0 & 1 & w_1^{(1)} \\ 2 & w_{-1}^{(2)} & -2 & w_- & 1 & w_1^{(1)} & w_1^{(2)} & 0 & w_0 & -1 & w_{-1}^{(1)} \end{pmatrix},$$

$$A_b := \begin{pmatrix} -2 & w_1^{(2)} & 2 & w_- & -1 & w_{-1}^{(1)} & w_{-1}^{(2)} & 0 & w_0 & 1 & w_1^{(1)} \\ 2 & w_- & 1 & w_1^{(1)} & w_{-1}^{(2)} & -2 & w_1^{(2)} & 0 & w_0 & -1 & w_{-1}^{(1)} \end{pmatrix}.$$

Consider the deep cycle $\mathcal{C}_t$ of top type through A_t. The losers of the arrows in this cycle are the letters in $w_1^{(2)}\ 0\ w_0\ \ -1\ w_{-1}^{(1)}$. The vertices corresponding to letters in $w_1^{(2)}$ or $-1\, w_{-1}^{(1)}$ are linked to some standard vertex. on the other hand, letters in $\{0\} \cup w_0$ give rise to free vertices. Writing $w_0 = w_0^{(1)} w_0^{(2)}$, these vertices are

$$F_t := \begin{pmatrix} -2 & w_- & -1 & w_{-1}^{(1)} & w_1^{(2)} & 2 & w_{-1}^{(2)} & 0 & w_0^{(1)} & w_0^{(2)} & 1 & w_1^{(1)} \\ 2 & w_{-1}^{(2)} & -2 & w_- & 1 & w_1^{(1)} & w_0^{(2)} & -1 & w_{-1}^{(1)} & w_1^{(2)} & 0 & w_0^{(1)} \end{pmatrix}.$$

Similarly, the deep cycle $\mathcal{C}_b$ of bottom type through A_b contains free vertices of the form

$$F_b := \begin{pmatrix} -2 & w_1^{(2)} & 2 & w_- & -1 & w_{-1}^{(1)} & w_0^{(2)} & 1 & w_1^{(1)} & w_{-1}^{(2)} & 0 & w_0^{(1)} \\ 2 & w_- & 1 & w_1^{(1)} & w_{-1}^{(2)} & -2 & w_1^{(2)} & 0 & w_0^{(1)} & w_0^{(2)} & -1 & w_{-1}^{(1)} \end{pmatrix}.$$

14.6. **Free vertices.** I claim that there are no other free vertices than those obtained in the last subsection. Consider the deep bottom cycle Ξ_b through F_t. This cycle is actually *ultradeep* in the sense that none of its vertices is linked to a standard vertex. On the other hand, given any vertex in Ξ_b, the top cycle through it contains vertices which are linked to some standard vertex (it is sufficient to have $\alpha_b = -1$). This proves the claim.

What we have just proved is that there are two types of deep cycles: cycles of depth 1 which contain at least one vertex linked to a standard vertex and cycles of depth 2 which do not contain such a vertex. All free vertices belong to two deep cycles, one of depth 1 and one of depth 2. This allow to separate the free vertices into top and bottom type, according to the type of the deep cycle of depth 1 through them. The two types are exchanged by the involution.

To count the free vertices of top type, observe that F_t is uniquely determined by the 6 words $w_-, w_0^{(1)}, w_0^{(2)}, w_1^{(1)}, w_{-1}^{(2)}, w_{-1}^{(1)} w_1^{(2)}$. The fact that this does not allow to determine w_1, w_{-1} reflects the fact that the deep cycle $\mathcal{C}_t$ through F_t contains vertices linked to standard vertices $\neq S(w)$. As the sum of the lengths of these six words is equal to N, the number of free vertices of top type is

$$N! \sum_{n_1 + \ldots n_6 = N} 1 = \frac{1}{120}(N+5)!\ .$$

14.7. **Number of vertices.** From the previous computations, one gets

$$N(\mathcal{D}) = (N^2 + 7N + 13) N_{st}(\mathcal{D}) - 2\delta(\mathcal{D}) + \frac{1}{60}(N+5)! = \frac{1}{8}(N+5)!.$$

15. The diagrams $[5+N,2](0)(1^2,0^{N-1})$

15.1. **Alphabet, Automorphisms, Involution.** The alphabet is

$$\mathcal{A} := \{-1, 1\} \sqcup \{a, b, c, d\} \sqcup \mathcal{A}^*,$$

where $\mathcal{A}^*$ has $(N-1)$ letters.

The automorphism group is the product of the cyclic group of order 4, permuting cyclically a, b, c, d, and the permutation group of $\mathcal{A}^*$.

There are two involutions. The involution I_0 exchanges -1 and 1, a and c and fixes b,d. The involution I_1 exchanges -1 and 1, b and d and fixes a,c.

15.2. **Standard vertices.** They are parametrized by a letter $x \in \{a, b, c, d\}$ and a symbol $w = (w_-, w_a, w_b, w_c, w_d)$.

$$S_a(w) := \begin{pmatrix} -1 & w_- & b & w_b & c & w_c & d & w_d & a & w_a & 1 \\ 1 & w_- & d & w_d & c & w_c & b & w_b & a & w_a & -1 \end{pmatrix}.$$

The number of standard vertices is

$$N_{st}(\mathcal{D}) = \frac{1}{6}(N+3)!.$$

15.3. **Edges of $\Gamma(\mathcal{D})$.** The edges from a standard vertex (here, $S_a(w)$) are associated to pair (α, β) of letters ordered in the same way by π_t and π_b.

(1) $\alpha, \beta \in w_-$: Write $w_- = w_-^{(1)} w_-^{(2)} w_-^{(3)}$. We have

$$x = a, \quad w'_- = w_-^{(2)} w_-^{(1)} w_-^{(3)}, \quad w'_b = w_b, \quad w'_c = w_c, \quad w'_d = w_d, \quad w'_a = w_a.$$

(2) $\alpha, \beta \in \{b\} \cup w_b$: Write $w_b = w_b^{(1)} w_b^{(2)} w_b^{(3)}$. We have

$$x = a, \quad w'_- = w_b^{(2)} w_-, \quad w'_b = w_b^{(1)} w_b^{(3)}, \quad w'_c = w_c, \quad w'_d = w_d, \quad w'_a = w_a.$$

(3) $\alpha, \beta \in \{c\} \cup w_c$: Write $w_c = w_c^{(1)} w_c^{(2)} w_c^{(3)}$. We have

$$x = a, \quad w'_- = w_c^{(2)} w_-, \quad w'_b = w_b, \quad w'_c = w_c^{(1)} w_c^{(3)}, \quad w'_d = w_d, \quad w'_a = w_a.$$

(4) $\alpha, \beta \in \{d\} \cup w_d$: Write $w_d = w_d^{(1)} w_d^{(2)} w_d^{(3)}$. We have

$$x = a, \quad w'_- = w_d^{(2)} w_-, \quad w'_b = w_b, \quad w'_c = w_c, \quad w'_d = w_d^{(1)} w_d^{(3)}, \quad w'_a = w_a.$$

(5) $\alpha, \beta \in \{a\} \cup w_a$: Write $w_a = w_a^{(1)} w_a^{(2)} w_a^{(3)}$. We have

$$x = a, \quad w'_- = w_a^{(2)} w_-, \quad w'_b = w_b, \quad w'_c = w_c, \quad w'_d = w_d, \quad w'_a = w_a^{(1)} w_a^{(3)}.$$

(6) $\alpha \in w_-, \beta \in \{b\} \cup w_b$:Write $w_- = w_-^{(1)} w_-^{(2)}$, $w_b = w_b^{(1)} w_b^{(2)}$. We have

$$x = a, \quad w'_- = w_-^{(2)}, \quad w'_b = w_b^{(1)} w_-^{(1)} w_b^{(2)}, \quad w'_c = w_c, \quad w'_d = w_d, \quad w'_a = w_a.$$

(7) $\alpha \in w_-, \beta \in \{c\} \cup w_c$: Write $w_- = w_-^{(1)} w_-^{(2)}$, $w_c = w_c^{(1)} w_c^{(2)}$. We have

$$x = a, \quad w'_- = w_-^{(2)}, \quad w'_b = w_b, \quad w'_c = w_c^{(1)} w_-^{(1)} w_c^{(2)}, \quad w'_d = w_d, \quad w'_a = w_a.$$

(8) $\alpha \in w_-, \beta \in \{d\} \cup w_d$: Write $w_- = w_-^{(1)} w_-^{(2)}$, $w_d = w_d^{(1)} w_d^{(2)}$. We have

$$x = a, \quad w'_- = w_-^{(2)}, \quad w'_b = w_b, \quad w'_c = w_c, \quad w'_d = w_d^{(1)} w_-^{(1)} w_d^{(2)}, \quad w'_a = w_a.$$

(9) $\alpha \in w_-, \beta \in \{a\} \cup w_a$: Write $w_- = w_-^{(1)} w_-^{(2)}$, $w_a = w_a^{(1)} w_a^{(2)}$. We have

$$x = a, \quad w'_- = w_-^{(2)}, \quad w'_b = w_b, \quad w'_c = w_c, \quad w'_d = w_d, \quad w'_a = w_a^{(1)} w_-^{(1)} w_a^{(2)}.$$

(10) $\alpha \in \{b\} \cup w_b, \beta \in \{a\} \cup w_a$: Write $w_b = w_b^{(1)} w_b^{(2)}$, $w_a = w_a^{(1)} w_a^{(2)}$. We have

$$x = b, \quad w'_- = w_b^{(2)}, \quad w'_b = w_b^{(1)} w_a^{(2)}, \quad w'_c = w_c, \quad w'_d = w_d, \quad w'_a = w_a^{(1)} w_-.$$

(11) $\alpha \in \{c\} \cup w_c, \beta \in \{a\} \cup w_a$: Write $w_c = w_c^{(1)} w_c^{(2)}$, $w_a = w_a^{(1)} w_a^{(2)}$. We have

$$x = c, \quad w'_- = w_c^{(2)}, \quad w'_b = w_b, \quad w'_c = w_c^{(1)} w_a^{(2)}, \quad w'_d = w_d, \quad w'_a = w_a^{(1)} w_-.$$

(12) $\alpha \in \{d\} \cup w_d, \beta \in \{a\} \cup w_a$: Write $w_d = w_d^{(1)} w_d^{(2)}$, $w_a = w_a^{(1)} w_a^{(2)}$. We have

$$x = d, \quad w'_- = w_d^{(2)}, \quad w'_b = w_b, \quad w'_c = w_c, \quad w'_d = w_d^{(1)} w_a^{(2)}, \quad w'_a = w_a^{(1)} w_-.$$

15.4. **Default of a vertex.** We have $\delta(S_a(w)) = \delta_1 + \delta_2 + \delta_3$, with

$$\begin{aligned}
\delta_1 &:= \frac{|w_-|(|w_-|-1)}{2} + \sum_{x=a,b,c,d} \frac{|w_x|(|w_x|+1)}{2}, \\
\delta_2 &:= |w_-|\,(4 + \sum_{x=a,b,c,d} |w_x|), \\
\delta_3 &:= (1+|w_a|)\,(3 + \sum_{x=b,c,d} |w_x|).
\end{aligned}$$

One obtains

$$\delta(S_a(w)) = \frac{N(N+1)}{2} + 2 + 2(|w_-| + |w_a|) - (|w_b||w_c| + |w_c||w_d| + |w_d||w_b|).$$

The default is maximal when w_b, w_c, w_d are empty. It is then equal to $\frac{N(N+5)}{2}$. The default is minimal when w_-, w_a are empty and

$$||w_b| - |w_c|| \leqslant 1, \quad ||w_c| - |w_d|| \leqslant 1, \quad ||w_d| - |w_b|| \leqslant 1.$$

It is then equal to $\lfloor \frac{(N+3)(N+4)}{6} \rfloor$.

The proof that $\Gamma(\mathcal{D})$ is connected is as in the previous sections. This implies that the list of standard vertices is as stated.

15.5. **Default of the diagram.** One gets

$$\begin{aligned}
\delta(\mathcal{D}) &= \frac{1}{2} \sum_{x,w} \delta(S_x(w)) \\
&= 2 \sum_w \delta(S_a(w)) \\
&= (N^2 + N + 4) \sum_w 1 + 8 \sum_w |w_-| - 6 \sum_w |w_b||w_c|.
\end{aligned}$$

Here on has

$$\begin{aligned}
\sum_w 1 &= \frac{(N+3)!}{4!}, \\
\sum_w |w_-| &= (N-1)\frac{(N+3)!}{5!}, \\
\sum_w |w_b||w_c| &= (N-1)(N-2)\frac{(N+3)!}{6!}.
\end{aligned}$$

One thus obtains

$$\delta(\mathcal{D}) = (4N^2 + 16N + 10)\frac{(N+3)!}{5!}.$$

15.6. **Open linked vertices.** Unconstrained linked vertices to $S_a(w)$ are obtained from three types of pairs (α, β) of letters of $\mathcal{A}$.

(1) $\alpha \in \{b\} \cup w_b, \beta \in \{c\} \cup w_c$: Write $w_b = w_b^{(1)} w_b^{(2)}$, $w_c = w_c^{(1)} w_c^{(2)}$. The two vertices obtained from (α, β) are

$$\begin{pmatrix} -1 & w_- & b & w_b^{(1)} & w_c^{(2)} & d & w_d & a & w_a & 1 & w_b^{(2)} & c & w_c^{(1)} \\ 1 & w_b^{(2)} & a & w_a & -1 & w_- & d & w_d & c & w_c^{(1)} & w_c^{(2)} & b & w_b^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} -1 & w_c^{(2)} & d & w_d & a & w_a & 1 & w_- & b & w_b^{(1)} & w_b^{(2)} & c & w_c^{(1)} \\ 1 & w_- & d & w_d & c & w_c^{(1)} & w_b^{(2)} & a & w_a & -1 & w_c^{(2)} & b & w_b^{(1)} \end{pmatrix}.$$

There is a deep cycle of top type through the first vertex but all the vertices in this cycle are linked to some standard vertex.

Similarly, there is a deep cycle of bottom type through the second vertex but all the vertices in this cycle are linked to some standard vertex.

(2) $\alpha \in \{d\} \cup w_d, \beta \in \{c\} \cup w_c$: This case is similar, applying the involution I_1.

(3) $\alpha \in \{d\} \cup w_d, \beta \in \{b\} \cup w_b$: Write $w_b = w_b^{(1)} w_b^{(2)}$, $w_d = w_d^{(1)} w_d^{(2)}$. The two vertices obtained from (α, β) are

$$A_t := \begin{pmatrix} -1 & w_- & b & w_b^{(1)} & w_d^{(2)} & a & w_a & 1 & w_b^{(2)} & c & w_c & d & w_d^{(1)} \\ 1 & w_b^{(2)} & a & w_a & -1 & w_- & d & w_d^{(1)} & w_d^{(2)} & c & w_c & b & w_b^{(1)} \end{pmatrix},$$

$$A_b := \begin{pmatrix} -1 & w_d^{(2)} & a & w_a & 1 & w_- & b & w_b^{(1)} & w_b^{(2)} & c & w_c & d & w_d^{(1)} \\ 1 & w_- & d & w_d^{(1)} & w_b^{(2)} & a & w_a & -1 & w_d^{(2)} & c & w_c & b & w_b^{(1)} \end{pmatrix}.$$

Consider the deep cycle $\mathcal{C}_t$ of top type through A_t. The losers of the arrows in this cycle are the letters in $w_d^{(2)}\ c\ w_c\ b\ w_b^{(1)}$. The vertices corresponding to letters in $w_d^{(2)}$ or $b\, w_b^{(1)}$ are linked to some standard vertex. on the other hand, letters in $\{c\} \cup w_c$ give rise to free vertices. Writing $w_c = w_c^{(1)} w_c^{(2)}$, these vertices F_t are

$$\begin{pmatrix} -1 & w_- & b & w_b^{(1)} & w_d^{(2)} & a & w_a & 1 & w_b^{(2)} & c & w_c^{(1)} & w_c^{(2)} & d & w_d^{(1)} \\ 1 & w_b^{(2)} & a & w_a & -1 & w_- & d & w_d^{(1)} & w_c^{(2)} & b & w_b^{(1)} & w_d^{(2)} & c & w_c^{(1)} \end{pmatrix}.$$

Applying I_1, we get also vertices F_b.

15.7. **Free vertices.** The discussion is similar to the last section.

We prove that there are no other free vertices than those obtained in the last subsection. Consider the deep bottom cycle Ξ_b through F_t. This cycle has depth 2. On the other hand, given any vertex in Ξ_b, the top cycle through it contains vertices which are linked to some standard vertex (it is sufficient to have $\alpha_b = b$). This proves the assertion.

All free vertices belong to two deep cycles, one of depth 1 and one of depth 2. This allow to separate the free vertices into top and bottom type, according to the type of the deep cycle of depth 1 through them. The two types are exchanged by the involution.

To count the free vertices of top type, observe that F_t is uniquely determined by the 7 words $w_-, w_a, w_c^{(1)}, w_c^{(2)}, w_d^{(1)}, w_b^{(2)}, w_b^{(1)} w_d^{(2)}$. The fact that this does not allow to determine w_b, w_d reflects the fact that the deep cycle $\mathcal{C}_t$ through F_t contains vertices linked to standard vertices $\neq S_a(w)$. As the sum of the lengths of these 7 words is equal to $N-1$, the number of free vertices of top type is

$$4(N-1)! \sum_{n_1+\dots n_7=N-1} 1 = \frac{1}{180}(N+5)!\ .$$

15.8. **Number of vertices.** From the previous computations, one gets

$$N(\mathcal{D}) = (N^2+7N+13)N_{st}(\mathcal{D}) - 2\delta(\mathcal{D}) + \frac{1}{90}(N+5)! = \frac{1}{9}(N+5)!.$$

16. THE DIAGRAMS $[6+N,3](4)(0^N)hyp$

16.1. **Alphabet, automorphisms and involution.** The alphabet is $\mathcal{A} = \{\pm5,\pm3,\pm1\} \sqcup \mathcal{A}^*$, where $\mathcal{A}^*$ has N letters. The involution fixes each letter in $\mathcal{A}^*$, and exchanges $\pm1,\pm3,\pm5$. The automorphism group is the permutation group of $\mathcal{A}^*$.

16.2. **Standard vertices.** They are parametrized by a symbol $w = (w_-, w_{-3}, w_{-1}, w_1, w_3)$. We will write W_i for $i\,w_i$, $i=-3,-1,1,3$. This is necessary as the diagrams are getting more complicated. The standard vertex $S(w)$ is

$$S(w) := \begin{pmatrix} -5 & w_- & -3 & w_{-3} & -1 & w_{-1} & 1 & w_1 & 3 & w_3 & 5 \\ 5 & w_- & 3 & w_3 & 1 & w_1 & -1 & w_{-1} & -3 & w_{-3} & -5 \end{pmatrix},$$

that we rewrite as

$$S(w) = \begin{pmatrix} -5 & w_- & W_{-3} & W_{-1} & W_1 & W_3 & 5 \\ 5 & w_- & W_3 & W_1 & W_{-1} & W_{-3} & -5 \end{pmatrix}.$$

The number of standard vertices is given by

$$N_{st}(\mathcal{D}) = N! \sum_{n_0+\dots n_4=N} 1 = \frac{1}{24}(N+4)!.$$

16.3. **Edges of $\Gamma(\mathcal{D})$.** The edges from a standard vertex (here, $S_a(w)$) are associated to pair (α,β) of letters ordered in the same way by π_t and π_b.

(1) $\alpha,\beta \in w_-$: We write $w_- = w_-^{(1)}w_-^{(2)}w_-^{(3)}$ and have

$$w'_- = w_-^{(2)}w_-^{(1)}w_-^{(3)}, \quad W'_{-3} = W_{-3}, \quad W'_{-1} = W_{-1}, \quad W'_1 = W_1, \quad W'_3 = W_3.$$

(2) $\alpha,\beta \in W_{-3}$: We write $W_{-3} = W_{-3}^{(1)}W_{-3}^{(2)}W_{-3}^{(3)}$ and have

$$w'_- = W_{-3}^{(2)}w_-, \quad W'_{-3} = W_{-3}^{(1)}W_{-3}^{(3)}, \quad W'_{-1} = W_{-1}, \quad W'_1 = W_1, \quad W'_3 = W_3.$$

(3) $\alpha,\beta \in W_{-1}$: We write $W_{-1} = W_{-1}^{(1)}W_{-1}^{(2)}W_{-1}^{(3)}$ and have

$$w'_- = W_{-1}^{(2)}w_-, \quad W'_{-3} = W_{-3}, \quad W'_{-1} = W_{-1}^{(1)}W_{-1}^{(3)}, \quad W'_1 = W_1, \quad W'_3 = W_3.$$

(4) $\alpha,\beta \in W_1$: We write $W_1 = W_1^{(1)}W_1^{(2)}W_1^{(3)}$ and have

$$w'_- = W_1^{(2)}w_-, \quad W'_{-3} = W_{-3}, \quad W'_{-1} = W_{-1}, \quad W'_1 = W_1^{(1)}W_1^{(3)}, \quad W'_3 = W_3.$$

(5) $\alpha, \beta \in W_3$: We write $W_3 = W_3^{(1)} W_3^{(2)} W_3^{(3)}$ and have

$$w'_- = W_3^{(2)} w_-, \quad W'_{-3} = W_{-3}, \quad W'_{-1} = W_{-1}, \quad W'_1 = W_1, \quad W'_3 = W_3^{(2)} W_3^{(1)} W_3^{(3)}.$$

(6) $\alpha \in w_-, \beta \in W_{-3}$: We write $w_- = w_-^{(1)} w_-^{(2)}$, $W_{-3} = W_{-3}^{(1)} W_{-3}^{(2)}$ and have

$$w'_- = w_-^{(2)}, \quad W'_{-3} = W_{-3}^{(1)} w_-^{(1)} W_{-3}^{(2)}, \quad W'_{-1} = W_{-1}, \quad W'_1 = W_1, \quad W'_3 = W_3.$$

(7) $\alpha \in w_-, \beta \in W_{-1}$: We write $w_- = w_-^{(1)} w_-^{(2)}$, $W_{-1} = W_{-1}^{(1)} W_{-1}^{(2)}$ and have

$$w'_- = w_-^{(2)}, \quad W'_{-3} = W_{-3}, \quad W'_{-1} = W_{-1}^{(1)} w_-^{(1)} W_{-1}^{(2)}, \quad W'_1 = W_1, \quad W'_3 = W_3.$$

(8) $\alpha \in w_-, \beta \in W_1$: We write $w_- = w_-^{(1)} w_-^{(2)}$, $W_1 = W_1^{(1)} W_1^{(2)}$ and have

$$w'_- = w_-^{(2)}, \quad W'_{-3} = W_{-3}, \quad W'_{-1} = W_{-1}, \quad W'_1 = W_1^{(1)} w_-^{(1)} W_1^{(2)}, \quad W'_3 = W_3.$$

(9) $\alpha \in w_-, \beta \in W_3$: We write $w_- = w_-^{(1)} w_-^{(2)}$, $W_3 = W_3^{(1)} W_3^{(2)}$ and have

$$w'_- = w_-^{(2)}, \quad W'_{-3} = W_{-3}, \quad W'_{-1} = W_{-1}, \quad W'_1 = W_1, \quad W'_3 = W_3^{(1)} w_-^{(1)} W_3^{(2)}.$$

16.4. **Default of a vertex.** We have $\delta(S(w)) = \delta_1 + \delta_2$ with

$$\begin{aligned} \delta_1 &= \frac{|w_-|(|w_-|-1)}{2} + \sum_{x=-3,-1,1,3} \frac{|w_x|(|w_x|+1)}{2}, \\ \delta_2 &= |w_-| \, (4 + \sum_{x=-3,-1,1,3} |w_x|). \end{aligned}$$

One obtains

$$\delta(S_a(w)) = \frac{N(N+1)}{2} + 3|w_-| - \sum_{i,j \in \{-3,-1,1,3\}, i<j} |w_i||w_j|.$$

The default is maximal when w_i is empty for $i = -3, -1, 1, 3$. It is then equal to $\frac{N(N+7)}{2}$. The default is minimal when w_- is empty and

$$||w_i| - |w_j|| \leqslant 1, \quad \forall i, j \in \{-3, -1, 1, 3\}.$$

It is then equal to $\lfloor \frac{(N+2)^2}{8} \rfloor$.

The proof that $\Gamma(\mathcal{D})$ is connected is as in the previous sections. This implies that the list of standard vertices is as stated.

16.5. **Formulas frequently used.**

$$\sum_{n_0 + \ldots + n_k = N} 1 = \frac{(N+k)!}{k! \, N!}.$$

$$\sum_{n_0 + \ldots + n_k = N} n_0 = \frac{(N+k)!}{(k+1)!(N-1)!}.$$

$$\sum_{n_0 + \ldots + n_k = N} n_0 n_1 = \frac{(N+k)!}{(k+2)!(N-2)!}.$$

16.6. **Default of the diagram.**

$$\begin{aligned}\delta(\mathcal{D}) &= \frac{1}{2}\sum_{x,w}\delta(S_x(w))\\ &= \frac{N(N+1)}{4}\sum_w 1+\frac{3}{2}\sum_w |w_-| - 3\sum_w |w_1||w_{-1}|.\end{aligned}$$

From the formulas in the last subsection, one has

$$\begin{aligned}\sum_w 1 &= \frac{(N+4)!}{4!},\\ \sum_w |w_-| &= N\frac{(N+4)!}{5!},\\ \sum_w |w_1||w_{-1}| &= N(N-1)\frac{(N+4)!}{6!}.\end{aligned}$$

One thus obtains

$$\delta(\mathcal{D}) = N\,(3N+13)\frac{(N+4)!}{480}.$$

16.7. **Open linked vertices.** Unconstrained linked vertices to $S_a(w)$ are obtained from six types of pairs (α,β) of letters of $\mathcal{A}$.

(1) $\alpha\in W_{-3},\beta\in W_{-1}$: Write $W_{-3}=W_{-3}^{(1)}W_{-3}^{(2)}$, $W_{-1}=W_{-1}^{(1)}W_{-1}^{(2)}$. The two vertices obtained from (α,β) are

$$\begin{pmatrix} -5 & w_- & W_{-3}^{(1)} & W_{-1}^{(2)} & W_1 & W_3 & 5 & W_{-3}^{(2)} & W_{-1}^{(1)}\\ 5 & W_{-3}^{(2)} & -5 & w_- & W_3 & W_1 & W_{-1}^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)}\end{pmatrix},$$

$$\begin{pmatrix} -5 & W_{-1}^{(2)} & W_1 & W_3 & 5 & w_- & W_{-3}^{(1)} & W_{-3}^{(2)} & W_{-1}^{(1)}\\ 5 & w_- & W_3 & W_1 & W_{-1}^{(1)} & W_{-3}^{(2)} & -5 & W_{-1}^{(2)} & W_{-3}^{(1)}\end{pmatrix}.$$

The deep cycles through these vertices contain only linked vertices.

(2) $\alpha\in W_{-1},\beta\in W_1$: Write $W_{-1}=W_{-1}^{(1)}W_{-1}^{(2)}$, $W_1=W_1^{(1)}W_1^{(2)}$. The two vertices obtained from (α,β) are

$$\begin{pmatrix} -5 & w_- & W_{-3} & W_{-1}^{(1)} & W_1^{(2)} & W_3 & 5 & W_{-1}^{(2)} & W_1^{(1)}\\ 5 & W_{-1}^{(2)} & W_{-3} & -5 & w_- & W_3 & W_1^{(1)} & W_1^{(2)} & W_{-1}^{(1)}\end{pmatrix},$$

$$\begin{pmatrix} -5 & W_1^{(2)} & W_3 & 5 & w_- & W_{-3} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_1^{(1)}\\ 5 & w_- & W_3 & W_1^{(1)} & W_{-1}^{(2)} & W_{-3} & -5 & W_1^{(2)} & W_{-1}^{(1)}\end{pmatrix}.$$

The deep cycles through these vertices contain only linked vertices.

(3) $\alpha\in W_1,\beta\in W_3$: Write $W_1=W_1^{(1)}W_1^{(2)}$, $W_3=W_3^{(1)}W_3^{(2)}$. The two vertices obtained from (α,β) are

$$\begin{pmatrix} -5 & w_- & W_{-3} & W_{-1} & W_1^{(1)} & W_3^{(2)} & 5 & W_1^{(2)} & W_3^{(1)}\\ 5 & W_1^{(2)} & W_{-1} & W_{-3} & -5 & w_- & W_3^{(1)} & W_3^{(2)} & W_1^{(1)}\end{pmatrix},$$

$$\begin{pmatrix} -5 & W_3^{(2)} & 5 & w_- & W_{-3} & W_{-1} & W_1^{(1)} & W_1^{(2)} & W_3^{(1)} \\ 5 & w_- & W_3^{(1)} & W_1^{(2)} & W_{-1} & W_{-3} & 5 & W_3^{(2)} & W_1^{(1)} \end{pmatrix}.$$

The deep cycles through these vertices contain only linked vertices.

(4) $\alpha \in W_{-3}, \beta \in W_1$: Write $W_{-3} = W_{-3}^{(1)} W_{-3}^{(2)}$, $W_1 = W_1^{(1)} W_1^{(2)}$. The two vertices obtained from (α, β) are

$$\begin{pmatrix} -5 & w_- & W_{-3}^{(1)} & W_1^{(2)} & W_3 & 5 & W_{-3}^{(2)} & W_{-1} & W_1^{(1)} \\ 5 & W_{-3}^{(2)} & -5 & w_- & W_3 & W_1^{(1)} & W_1^{(2)} & W_{-1} & W_{-3}^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} -5 & W_1^{(2)} & W_3 & 5 & w_- & W_{-3}^{(1)} & W_{-3}^{(2)} & W_{-1} & W_1^{(1)} \\ 5 & w_- & W_3 & W_1^{(1)} & W_{-3}^{(2)} & -5 & W_1^{(2)} & W_{-1} & W_{-3}^{(1)} \end{pmatrix}.$$

This gives rise to free vertices through the splitting of W_{-1}. See next subsection.

(5) $\alpha \in W_{-1}, \beta \in W_3$: Write $W_{-1} = W_{-1}^{(1)} W_{-1}^{(2)}$, $W_3 = W_3^{(1)} W_3^{(2)}$. The two vertices obtained from (α, β) are

$$\begin{pmatrix} -5 & w_- & W_{-3} & W_{-1}^{(1)} & W_3^{(2)} & 5 & W_{-1}^{(2)} & W_1 & W_3^{(1)} \\ 5 & W_{-1}^{(2)} & W_{-3} & -5 & w_- & W_3^{(1)} & W_3^{(2)} & W_1 & W_{-1}^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} -5 & W_3^{(2)} & 5 & w_- & W_{-3} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_1 & W_3^{(1)} \\ 5 & w_- & W_3^{(1)} & W_{-1}^{(2)} & W_{-3} & -5 & W_3^{(2)} & W_1 & W_{-1}^{(1)} \end{pmatrix}.$$

This gives rise to free vertices through the splitting of W_1. See next subsection.

(6) $\alpha \in W_{-3}, \beta \in W_3$: Write $W_{-3} = W_{-3}^{(1)} W_{-3}^{(2)}$, $W_3 = W_3^{(1)} W_3^{(2)}$. The two vertices obtained from (α, β) are

$$\begin{pmatrix} -5 & w_- & W_{-3}^{(1)} & W_3^{(2)} & 5 & W_{-3}^{(2)} & W_{-1} & W_1 & W_3^{(1)} \\ 5 & W_{-3}^{(2)} & -5 & w_- & W_3^{(1)} & W_3^{(2)} & W_1 & W_{-1} & W_{-3}^{(1)} \end{pmatrix},$$

$$\begin{pmatrix} -5 & W_3^{(2)} & 5 & w_- & W_{-3}^{(1)} & W_{-3}^{(2)} & W_{-1} & W_1 & W_3^{(1)} \\ 5 & w_- & W_3^{(1)} & W_{-3}^{(2)} & 5 & W_3^{(2)} & W_1 & W_{-1} & W_{-3}^{(1)} \end{pmatrix}.$$

This gives rise to free vertices through the splitting of W_1 or W_{-1}. See next subsection.

16.8. **Free vertices.** In the case (4) of last subsection, the splitting $W_{-1} = W_{-1}^{(1)} W_{-1}^{(2)}$ gives rise to the vertices

$$F_t := \begin{pmatrix} -5 & w_- & W_{-3}^{(1)} & W_1^{(2)} & W_3 & 5 & W_{-3}^{(2)} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_1^{(1)} \\ 5 & W_{-3}^{(2)} & -5 & w_- & W_3 & W_1^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)} & W_1^{(2)} & W_{-1}^{(1)} \end{pmatrix},$$

$$F_b := \begin{pmatrix} -5 & W_1^{(2)} & W_3 & 5 & w_- & W_{-3}^{(1)} & W_{-1}^{(2)} & W_1^{(1)} & W_{-3}^{(2)} & W_{-1}^{(1)} \\ 5 & w_- & W_3 & W_1^{(1)} & W_{-3}^{(2)} & -5 & W_1^{(2)} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)} \end{pmatrix}.$$

The free vertex F_t belongs to a deep cycle $\mathcal{C}_t$ of top type, depth 1 and to a deep cycle Ξ_b of bottom type depth 2. But all cycles of top type through a vertex of Ξ_b have depth 1.

The same holds for F_b, exchanging top and bottom.

The free vertices arising from case (5) of last subsection are dealt with symmetrically.

In case (6), the splitting $W_{-1} = W_{-1}^{(1)} W_{-1}^{(2)}$ gives rise to the vertices

$$G_t := \begin{pmatrix} -5 & w_- & W_{-3}^{(1)} & W_3^{(2)} & 5 & W_{-3}^{(2)} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_1 & W_3^{(1)} \\ 5 & W_{-3}^{(2)} & -5 & w_- & W_3^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)} & W_3^{(2)} & W_1 & W_{-1}^{(1)} \end{pmatrix},$$

$$G_b := \begin{pmatrix} -5 & W_3^{(2)} & 5 & w_- & W_{-3}^{(1)} & W_{-1}^{(2)} & W_1 & W_3^{(1)} & W_{-3}^{(2)} & W_{-1}^{(1)} \\ 5 & w_- & W_3^{(1)} & W_{-3}^{(2)} & -5 & W_3^{(2)} & W_1 & W_{-1}^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)} \end{pmatrix}.$$

The splitting $W_1 = W_1^{(1)} W_1^{(2)}$ is symmetric w.r.t. the involution.

From now on, we use the depth as defined in the first section[26].

The free vertices G_t, G_b have depth 6. Consider a vertex H_b of the bottom cycle Ξ_b through G_t. If the last top letter of H_b belongs to $W_{-1}^{(2)}$ or $W_3^{(1)}$, the depth of H_b is equal to 6. If on the other hand the last top letter belongs to W_1, we split $W_1 = W_1^{(1)} W_1^{(2)}$ and have

$$H_b := \begin{pmatrix} -5 & w_- & W_{-3}^{(1)} & W_3^{(2)} & 5 & W_{-3}^{(2)} & W_{-1}^{(1)} & W_1^{(2)} & W_3^{(1)} & W_{-1}^{(2)} & W_1^{(1)} \\ 5 & W_{-3}^{(2)} & -5 & w_- & W_3^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)} & W_3^{(2)} & W_1^{(1)} & W_1^{(2)} & W_{-1}^{(1)} \end{pmatrix}.$$

Now that all W_i have split, I claim that the depth of H_b is 8. One uses the method of the first section. One obtains actually[27] $\mathfrak{D}_t(H_b) = 9$ and $\mathfrak{D}_b(H_b) = 7$. This indicates that the top cycle Θ_t through H_b has depth 9, while the bottom cycle Ξ_b has depth 7. It remains to see that all vertices of Θ_t have depth 8, actually $\mathfrak{D}_t = 7$[28]. This is clear. With respect to H_b, the other vertices of Θ_t differ only by a circular permutation of the letters in the final words $W_1^{(2)} W_{-1}^{(1)}$ of the bottom line of H_b, and this does not alter the computation of $\mathfrak{D}_t$ and $\mathfrak{D}_b$.

The same does not happen for G_b: all vertices of the top cycle Ξ_t through G_b have length 6, because it is not possible to split W_1.

On the other hand, we could have decided to first split W_1; we would have obtained G'_t (similar to G_b) and G'_b (similar to G_t) giving rise to H_t of depth 8.

Therefore, up to the involution, every free vertex has been obtained in the discussion above. We recapitulate (with a slightly different, obvious, notation which allows to relate easily to the hyperelliptic case $N = 0$)

(1)

$$\begin{pmatrix} W(-5) & W(-3) & W(3) & W(5) & W_{-1}^{(1)} & W_{-1}^{(2)} & W_1^{(1)} \\ W(5) & W(-5) & W(3) & W_1^{(1)} & W_{-1}^{(2)} & W(-3) & W_{-1}^{(1)} \end{pmatrix},$$

[26] *Depth* corresponds to *height* in the first section.

[27] This notation doesn't appear anywhere else. Clearly, $\mathfrak{D}_t(H_b)$ is the depth/height of the top cycle through H_b.

[28] The meaning of $\mathfrak{D}_t$ is unclear.

(2)

$$\begin{pmatrix} W(-5) & W(3) & W(5) & W^{(1)}_{-3} & W^{(2)}_{-1} & W(1) & W^{(1)}_{-1} \\ W(5) & W(3) & W(1) & W(-5) & W^{(1)}_{-1} & W^{(2)}_{-1} & W^{(1)}_{-3} \end{pmatrix},$$

(3)

$$\begin{pmatrix} W(-5) & W(-3) & W(-1) & W(5) & W^{(1)}_{1} & W^{(2)}_{1} & W^{(1)}_{3} \\ W(5) & W(-3) & W(-5) & W^{(1)}_{3} & W^{(2)}_{1} & W(-1) & W^{(1)}_{1} \end{pmatrix},$$

(4)

$$\begin{pmatrix} W(-5) & W(5) & W(-3) & W^{(1)}_{-1} & W^{(2)}_{1} & W(3) & W^{(1)}_{1} \\ W(5) & W(3) & W(-3) & W(-5) & W^{(1)}_{1} & W^{(2)}_{1} & W^{(1)}_{-1} \end{pmatrix},$$

(5)

$$\begin{pmatrix} W(-5) & W(-3) & W(5) & W^{(1)}_{-1} & W^{(2)}_{-1} & W(1) & W^{(1)}_{3} \\ W(5) & W(-5) & W^{(1)}_{3} & W^{(2)}_{-1} & W(-3) & W(1) & W^{(1)}_{-1} \end{pmatrix},$$

(6)

$$\begin{pmatrix} W(-5) & W(5) & W^{(1)}_{-3} & W^{(2)}_{-1} & W(1) & W(3) & W^{(1)}_{-1} \\ W(5) & W(3) & W(-5) & W(1) & W^{(1)}_{-1} & W^{(2)}_{-1} & W^{(1)}_{-3} \end{pmatrix},$$

(7)

$$\begin{pmatrix} W(-5) & W(-3) & W(5) & W(-1) & W^{(1)}_{1} & W^{(2)}_{1} & W^{(1)}_{3} \\ W(5) & W(-5) & W^{(1)}_{3} & W^{(2)}_{1} & W(-1) & W(-3) & W^{(1)}_{1} \end{pmatrix},$$

(8)

$$\begin{pmatrix} W(-5) & W(5) & W^{(1)}_{-3} & W^{(2)}_{1} & W(3) & W(-1) & W^{(1)}_{1} \\ W(5) & W(3) & W(-5) & W^{(1)}_{1} & W^{(2)}_{1} & W(-1) & W^{(1)}_{-3} \end{pmatrix},$$

(9)

$$\begin{pmatrix} W(-5) & W(-3) & W(5) & W^{(1)}_{-1} & W^{(2)}_{1} & W(3) & W^{(1)}_{1} \\ W(5) & W(-5) & W(3) & W(-3) & W^{(1)}_{1} & W^{(2)}_{1} & W^{(1)}_{-1} \end{pmatrix},$$

(10)

$$\begin{pmatrix} W(-5) & W(5) & W(-3) & W(3) & W^{(1)}_{-1} & W^{(2)}_{-1} & W^{(1)}_{1} \\ W(5) & W(3) & W(-5) & W^{(1)}_{1} & W^{(2)}_{-1} & W(-3) & W^{(1)}_{-1} \end{pmatrix}.$$

The first eight categories of free vertices have depth 6 and the last two have depth 8.

In each category, the number of free vertices is

$$N! \sum_{n_0+n_1+\ldots+n_6=N} 1 = \frac{(N+6)!}{6!}.$$

16.9. **The total number of vertices.** From the previous computations, one gets

$$N(\mathcal{D}) = (N^2 + 9N + 21)N_{st}(\mathcal{D}) - 2\delta(\mathcal{D}) + \frac{1}{72}(N+6)! = \frac{31}{720}(N+6)!.$$

The counting can be separated according to the depth.

$$\begin{aligned}
N_0(\mathcal{D}) &= N_{st}(\mathcal{D}) = \frac{(N+4)!}{4!},\\
N_2(\mathcal{D}) &= (2N+10)N_{st}(\mathcal{D}) = \frac{(N+4)!(N+4)}{12},\\
N_4(\mathcal{D}) &= \frac{(7N^2+57N+120)(N+4)!}{240},\\
N_6(\mathcal{D}) &= \frac{1}{90}(N+6)!,\\
N_8(\mathcal{D}) &= \frac{1}{360}(N+6)!.
\end{aligned}$$

17. THE DIAGRAMS $[6+N,3](4)(0^N)$ *odd*

17.1. **Alphabet, automorphisms, involution.** The alphabet is $\mathcal{A} = \mathcal{A}_6 \sqcup \mathcal{A}^*$, where $\mathcal{A}^*$ has N letters. The involution fixes each letter in $\mathcal{A}^*$ and is the usual involution on $\mathcal{A}_6$. The automorphism group is the permutation group of $\mathcal{A}^*$.

17.2. **Standard vertices.** Recall from a previous section the 7 standard vertices in the case $N = 0$:

$$S := \begin{pmatrix} -5 & -3 & 3 & -1 & 1 & 5 \\ 5 & 3 & -3 & 1 & -1 & -5 \end{pmatrix},$$

which is fixed by the involution.

The others 6 come into 3 pairs of symmetric vertices

$$A^+ := \begin{pmatrix} -5 & 3 & -1 & 1 & -3 & 5 \\ 5 & 1 & 3 & -3 & -1 & -5 \end{pmatrix}, \quad A^- := \begin{pmatrix} -5 & -1 & -3 & 3 & 1 & 5 \\ 5 & -3 & 1 & -1 & 3 & -5 \end{pmatrix},$$
$$B^+ := \begin{pmatrix} -5 & 3 & -1 & -3 & 1 & 5 \\ 5 & 1 & -1 & 3 & -3 & -5 \end{pmatrix}, \quad B^- := \begin{pmatrix} -5 & -1 & 1 & -3 & 3 & 5 \\ 5 & -3 & 1 & 3 & -1 & -5 \end{pmatrix},$$
$$C^+ := \begin{pmatrix} -5 & 3 & 1 & -1 & -3 & 5 \\ 5 & 1 & -3 & -1 & 3 & -5 \end{pmatrix}, \quad C^- := \begin{pmatrix} -5 & -1 & 3 & 1 & -3 & 5 \\ 5 & -3 & -1 & 1 & 3 & -5 \end{pmatrix}.$$

It is thus reasonable to expect 7 families of standard vertices, each parametrized by a symbol $w = (w_-, w_{-3}, w_{-1}, w_1, w_3)$. For instance

$$A^+(w) := \begin{pmatrix} -5 & w_- & W_3 & W_{-1} & W_1 & W_{-3} & 5 \\ 5 & w_- & W_1 & W_3 & W_{-3} & W_{-1} & -5 \end{pmatrix},$$

where $W_i = i\ w_i$ for $i = \pm 1, \pm 3$.

The number of standard vertices is

$$N_{st}(\mathcal{D}) = 7\frac{(N+4)!}{4!}.$$

17.3. **Edges of** $\Gamma(\mathcal{D})$**.** The edges from a standard vertex are associated to pair (α, β) of letters ordered in the same way by π_t and π_b.

The first type of edges correspond to α, β belonging to the same subset (w_- or W_i). The vertices linked by such an edge belong to the same family. The case where $\alpha, \beta \in w_-$ gives edges allowing to rearrange w_- while the other four cases allow to transfer the end of W_i to the beginning of w_-. Viewed from the other endpoint, this corresponds to the cases $\alpha \in w_-, \beta \in W_i$.

The cases where $\alpha \in W_i, \beta \in W_j$ with $i \neq j$ correspond to edges whose endpoints lie in distinct families. There are 9 such possibilities:

- S and A^+ are related through (W_{-3}, W_1);
- S and B^+ are related through (W_{-3}, W_{-1});
- A^+ and B^- are related through (W_{-3}, W_3);
- A^+ and C^- are related through (W_3, W_{-1});
- C^+ and C^- are related through (W_1, W_{-1});
- the other cases are obtained from the involution.

The proof that the list of standard vertices is correct is as usual.

17.4. **Default of a standard vertex.** We have to treat each family separately. Because of the involution there are really 4 cases. However the dependence on the family affects only the last term in the sum $\delta_1 + \delta_2 + \delta_3$.

We have

$$\begin{aligned} \delta_1 &= \frac{|w_-|(|w_-|-1)}{2} + \sum_{x=-3,-1,1,3} \frac{|w_x|(|w_x|+1)}{2}, \\ \delta_2 &= |w_-|\,(4 + \sum_{x=-3,-1,1,3} |w_x|). \end{aligned}$$

We have seen in the hyperelliptic case that

$$\delta_1 + \delta_2 = \frac{N(N+1)}{2} + 3|w_-| - \sum_{i,j \in \{-3,-1,1,3\}, i<j} |w_i||w_j|.$$

Regarding δ_3, we have

- For a standard vertex in the S family,

$$\delta_3 = \sum_{i=\pm 3} \sum_{j=\pm 1} (1+|w_i|)(1+|w_j|).$$

  This gives

$$\delta(S(w)) = \frac{(N+2)(N+3)}{2} + 1 + |w_-| - (|w_3||w_{-3}| + |w_1||w_{-1}|).$$

- For a standard vertex in the A^+ family,

$$\delta_3 = (1+|w_3|)(1+|w_{-3}|) + (1+|w_3|)(1+|w_{-1}|) + (1+|w_1|)(1+|w_{-3}|).$$

  This gives

$$\begin{aligned}\delta(A^+(w)) &= \frac{(N+1)(N+2)}{2} + 2 + 2|w_-| + |w_3| + |w_{-3}| \\ &\quad - (|w_1||w_{-1}| + |w_1||w_3| + |w_{-3}||w_{-1}|).\end{aligned}$$

- For a standard vertex in the B^+ family,

$$\delta_3 = (1+|w_3|)(1+|w_{-3}|) + (1+|w_{-3}|)(1+|w_{-1}|).$$

This gives

$$\begin{aligned}\delta(B^+(w)) &= \frac{N(N+1)}{2} + 2 + 3|w_-| + |w_3| + 2|w_{-3}| + |w_{-1}| \\ &\quad - (|w_1||w_{-1}| + |w_1||w_3| + |w_{-3}||w_1| + |w_{-1}||w_3|).\end{aligned}$$

- For a standard vertex in the C^+ family,

$$\delta_3 = (1+|w_1|)(1+|w_{-1}|) + (1+|w_1|)(1+|w_{-3}|).$$

This gives

$$\begin{aligned}\delta(C^+(w)) &= \frac{N(N+1)}{2} + 2 + 3|w_-| + |w_{-3}| + 2|w_1| + |w_{-1}| \\ &\quad - (|w_3||w_{-3}| + |w_1||w_3| + |w_{-3}||w_{-1}| + |w_{-1}||w_3|).\end{aligned}$$

We do not discuss the mimimal values of the default. The maximal values are $\frac{N(N+7)}{2}+4$ for the S family, $\frac{N(N+7)}{2}+3$ for the A families, $\frac{N(N+7)}{2}+2$ for the B or C families.

17.5. **Default of the diagram.** We first sum the defaults over each family:

$$\begin{aligned}\frac{1}{2}\sum_w \delta(S(w)) &= \frac{N^2+5N+8}{4}\sum_w 1 + \frac{1}{2}\sum_w |w_-| - \sum_w |w_1||w_{-1}| \\ &= \frac{13N^2+83N+120}{2}\frac{(N+4)!}{6!}.\end{aligned}$$

$$\begin{aligned}\sum_w \delta(A^+(w)) &= \frac{N^2+3N+6}{2}\sum_w 1 + 4\sum_w |w_-| - 3\sum_w |w_1||w_{-1}| \\ &= (2N^2+12N+15)\frac{(N+4)!}{5!}.\end{aligned}$$

$$\begin{aligned}\sum_w \delta(B^+(w)) &= \sum_w \delta(C^+(w)) \\ &= \frac{N^2+N+4}{2}\sum_w 1 + 7\sum_w |w_-| - 4\sum_w |w_1||w_{-1}| \\ &= (11N^2+61N+60)\frac{(N+4)!}{6!}.\end{aligned}$$

The final result is

$$\delta(\mathcal{D}) = \frac{27N^2+157N+180}{4}\frac{(N+4)!}{5!}.$$

17.6. **Open linked vertices.** By definition, the two pure cycles through an open linked vertex have depth 3 and 5. The vertices on the deep cycle of depth 5 have either depth 4 (and are open linked) or depth 6 (and are free).

The vertices which are open linked to a standard vertex π correspond to pairs (α, β) of letters (distinct from ± 5) which are **not** ordered in the same way by π_t, π_b (i.e those not associated to edges of $\Gamma(\mathcal{D})$.

For S, the two possibilities ($\alpha \in W_1, \beta \in W_{-1}$ and $\alpha \in W_3, \beta \in W_{-3}$) produce linked open vertices such that the cycles of depth 5 through them contain only vertices of depth 4.

For $A^{\pm}$, there are three possibilities but none of them give rise to free vertices.

For $B^{\pm}$, $C^{\pm}$, there are four possibilities each, but three of them are sterile.

The fertile possibilities are $\alpha \in W_3, \beta \in W_{-3}$ for C^+ and C^-, $\alpha \in W_3, \beta \in W_1$ for B^+ and $\alpha \in W_{-1}, \beta \in W_{-3}$ for B^-. Each allows to split W_{-1} (for B^+ and C^+) or W_1 (for B^- and C^-) to produce free vertices.

17.7. **Free vertices.** It is sufficient to look at the free vertices arising from $C^+, \alpha \in W_3, \beta \in W_{-3}$ and $B^+, \alpha \in W_3, \beta \in W_1$ because the others are symmetric w.r.t. the involution.

These free vertices are

- From $C^+, \alpha \in W_3, \beta \in W_{-3}$

$$F_t := \begin{pmatrix} -5 & w_- & W_3^{(1)} & W_{-3}^{(2)} & 5 & W_3^{(2)} & W_1 & W_{-1}^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)} \\ 5 & W_3^{(2)} & -5 & w_- & W_1 & W_{-3}^{(1)} & W_{-1}^{(2)} & W_3^{(1)} & W_{-3}^{(2)} & W_{-1}^{(1)} \end{pmatrix},$$

$$F_b := \begin{pmatrix} -5 & W_{-3}^{(2)} & 5 & w_- & W_3^{(1)} & W_{-1}^{(2)} & W_{-3}^{(1)} & W_3^{(2)} & W_1 & W_{-1}^{(1)} \\ 5 & w_- & W_1 & W_{-3}^{(1)} & W_3^{(2)} & -5 & W_{-3}^{(2)} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_3^{(1)} \end{pmatrix}.$$

- From $B^+, \alpha \in W_3, \beta \in W_1$

$$G_t := \begin{pmatrix} -5 & w_- & W_3^{(1)} & W_1^{(2)} & 5 & W_3^{(2)} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_{-3} & W_1^{(1)} \\ 5 & W_3^{(2)} & W_{-3} & -5 & w_- & W_1^{(1)} & W_{-1}^{(2)} & W_3^{(1)} & W_1^{(2)} & W_{-1}^{(1)} \end{pmatrix},$$

$$G_b := \begin{pmatrix} -5 & W_1^{(2)} & 5 & w_- & W_3^{(1)} & W_{-1}^{(2)} & W_{-3} & W_1^{(1)} & W_3^{(2)} & W_{-1}^{(1)} \\ 5 & w_- & W_1^{(1)} & W_3^{(2)} & W_{-3} & -5 & W_1^{(2)} & W_{-1}^{(1)} & W_{-1}^{(2)} & W_3^{(1)} \end{pmatrix}.$$

Observe that F_b and G_b are of the same type. Observe also that the symmetric of G_t w.r.t. the involution belong to the same type than G_t. So we have at this stage five types of free vertices: one containing F_t, one containing the symmetric of F_t w.r.t. the involution, one containing F_b and G_b, the symmetric family, and a last one containing G_t which is autosymmetric. We represent these families by their only element when $N = 0$.

We now check that there are no other free vertices by computing the depth of the pure cycles through F_t, F_b, G_t, G_b. For each of these vertices, one of the cycles has depth 5 and is the one that we have used to access these vertices.

All vertices of the bottom cycle through F_t are free vertices of the same family. This cycle has depth 7.

Similarly, all vertices of the top cycle through F_b (or G_b) are free vertices of the same family. This cycle has depth 7.

The same holds for the symmetric families.

Finally, the bottom cycle through G_t contains vertices of depth 4 (put -3 in last position on the top line). Hence this cycle has depth 5.

The proof that there are no other free vertices is complete.

Each of the five families of free vertices is parametrized by a decomposition of $\mathcal{A}^*$ into seven ordered subsets. Therefore, the total number of free vertices is

$$N_{free} = 5\frac{(N+6)!}{6!}.$$

17.8. **Number of vertices.** The previous computations give

$$\begin{aligned} N(\mathcal{D}) &= (N^2+9N+21)N_{st}(\mathcal{D}) - 2\delta(\mathcal{D}) + N_{free}(\mathcal{D}) \\ &= 7(N^2+9N+21)\frac{(N+4)!}{4!} - \frac{27N^2+157N+180}{2}\frac{(N+4)!}{5!} + 5\frac{(N+6)!}{6!} \\ &= 134\frac{(N+6)!}{6!}. \end{aligned}$$

18. The quasihyperelliptic diagrams $[2g+1, g](0)(2g-2)$ and $[2g+2, g](0)(g-1, g-1)$

18.1. **Alphabet, automorphisms, involutions.** Let d be the number of letters and $D := d-2$. The alphabet is the union of the cyclic group $\mathbb{Z}_D$ and two special letters $\pm\infty$ which are the first letters in the top and bottom lines. The automorphism group is $\mathbb{Z}_D$. For each $m \in \mathbb{Z}_D$ there is an involution I_m which exchanges $\pm\infty$, fixes m and exchanges $m \pm k$. When D is even, it also fixes $m + D/2$ and the involutions I_m and $I_{m+D/2}$ coincide.

18.2. **Standard vertices.** There are D standard vertices, indexed by $\mathbb{Z}_D$. The vertex S_m is

$$\begin{pmatrix} -\infty & m+1 & m+2 & \dots & m & +\infty \\ +\infty & m-1 & m-2 & \dots & m & -\infty \end{pmatrix}.$$

18.3. **Edges of Γ_D.** The pairs of letters (α, β) which are ordered in the same way (in S_0) by π_t and π_b are the pairs with $\alpha \in \mathbb{Z}_D \setminus \{0\}, \beta = 0$.

Such a pair provides an edge in $\Gamma(\mathcal{D})$ between S_0 and S_m.

Therefore $\Gamma(\mathcal{D})$ is the full graph on D vertices.

18.4. **Defaults.** The default of each vertex is equal to $(D-1)$. The default of the diagram is equal to

$$\delta(\mathcal{D}) = \frac{D(D-1)}{2}.$$

18.5. **Open linked vertices.** Let $\alpha < \beta$ be a pair of letters in $(\mathbb{Z}_D)^*$. The two open linked vertices obtained from this pair are

$$\begin{pmatrix} -\infty & 1 & \dots & \alpha & \beta+1 & \dots & 0 & +\infty & \alpha+1 & \dots & \beta \\ +\infty & \alpha-1 & \alpha-2 & \dots & 1 & 0 & -\infty & -1 & \dots & \alpha+1 & \alpha \end{pmatrix},$$

$$\begin{pmatrix} -\infty & \beta+1 & \beta+2 & \dots & -1 & 0 & +\infty & 1 & \dots & \beta-1 & \beta \\ +\infty & -1 & \dots & \beta & \alpha-1 & \dots & 0 & -\infty & \beta-1 & \dots & \alpha \end{pmatrix}.$$

The two vertices above will be abbreviated as

$$\begin{pmatrix} -\infty & (0 \nearrow \alpha] & (\beta \nearrow 0] & +\infty & (\alpha \nearrow \beta] \\ +\infty & (\alpha \searrow 0] & & -\infty & (0 \searrow \alpha] \end{pmatrix},$$
$$\begin{pmatrix} -\infty & (\beta \nearrow 0] & & +\infty & (0 \nearrow \beta] \\ +\infty & (0 \searrow \beta] & (\alpha \searrow 0] & -\infty & (\beta \searrow \alpha] \end{pmatrix}.$$

18.6. **Free vertices.** For α, β as above, let us choose γ with $\alpha < \gamma < \beta$ (when $\beta - \alpha > 1$; when $\beta - \alpha = 1$, we will not have associated free vertices).

We get a pair of free vertices of depth 6:

$$F_t := \begin{pmatrix} -\infty & (0 \nearrow \alpha] & (\beta \nearrow 0] & +\infty & (\alpha \nearrow \gamma] & (\gamma \nearrow \beta] \\ +\infty & (\alpha \searrow 0] & -\infty & (0 \searrow \beta] & (\gamma \searrow \alpha] & (\beta \searrow \gamma] \end{pmatrix},$$
$$F_b := \begin{pmatrix} -\infty & (\beta \nearrow 0] & +\infty & (0 \nearrow \alpha] & (\gamma \nearrow \beta] & (\alpha \nearrow \gamma] \\ +\infty & (0 \searrow \beta] & (\alpha \searrow 0] & -\infty & (\beta \searrow \gamma] & (\gamma \searrow \alpha] \end{pmatrix}.$$

Consider the bottom cycle Ξ_b through F_t. If $\beta = \gamma + 1$, the only vertex of Ξ_b is F_t , which is inessential. Otherwise, the other vertices of Ξ_b are parametrized by an element θ such that $\gamma < \theta < \beta$:

$$G_b := \begin{pmatrix} -\infty & (0 \nearrow \alpha] & (\beta \nearrow 0] & +\infty & (\alpha \nearrow \gamma] & (\theta \nearrow \beta] & (\gamma \nearrow \theta] \\ +\infty & (\alpha \searrow 0] & -\infty & (0 \searrow \beta] & (\gamma \searrow \alpha] & (\beta \searrow \theta] & (\theta \searrow \gamma] \end{pmatrix}.$$

These vertices have now depth 8 while Ξ_b has depth 7 (in all cases).

To understand the formation of theses vertices, it is better to change notations, using

$$\alpha_0 = \alpha, \quad \alpha_1 = \beta, \quad \alpha_2 = \gamma, \quad \alpha_3 = \theta, \dots$$

for vertices starting from F_t and

$$\alpha_0 = \beta, \quad \alpha_1 = \alpha, \quad \alpha_2 = \gamma, \dots$$

or vertices starting from F_b.

The α_i should satisfy

$$\alpha_0 < \alpha_2 < \dots < \alpha_{2n} < \dots < \alpha_{2n+1} < \dots < \alpha_3 < \alpha_1$$

in the first case and

$$\alpha_0 > \alpha_2 > \dots > \alpha_{2n} > \dots > \alpha_{2n+1} > \dots > \alpha_3 > \alpha_1$$

in the second case.

With this new notation, we have

$$F_t = F(\alpha, \beta, \gamma), \quad G_b = F(\alpha, \beta, \gamma, \theta), \quad F_b = (\beta, \alpha, \gamma).$$

The depth of a vertex $F(\alpha_0, \dots, \alpha_{n-1})$ is equal to $2n$. The two cycles through this vertex have depths equal to $2n-1$ and $2n+1$.

It is fundamental to observe that free vertices are accessible from a well-defined standard vertex! Only constrained linked vertices provide bridges between standard vertices.

By comparing with the pure hyperelliptic case, we see that the number of free vertices is

$$N_{free}(\mathcal{D}) = D(2^D - D(D-1) - 2).$$

18.7. **Number of vertices.** We have

$$\begin{aligned} N(\mathcal{D}) &= [D(D+1)+1]N_{st}(\mathcal{D}) - 2\delta(\mathcal{D}) + N_{free}(\mathcal{D}) \\ &= D(2^D + D). \end{aligned}$$

19. THE DIAGRAM $[7,3](0)(4)odd$

19.1. **Alphabet, Automorphisms and Involutions.** The alphabet is $\mathcal{A} = \{\pm\infty\} \sqcup \mathbb{Z}_5$. The automorphism group is $\mathbb{Z}_5$. There are five involutions, indexed by $\mathbb{Z}_5$. The involution I_m exchanges $\pm\infty$, fixes m and exchanges $m \pm k$.

19.2. **Standard vertices.** The standard vertices are indexed by an element of $\mathbb{Z}_5$ and a standard vertex of the diagram $[6,2](4)odd$ (recall that there are seven of them, S,$A^\pm$, $B^\pm$, $C^\pm$). For instance

$$A^+(0) = \begin{pmatrix} -\infty & 2 & 1 & -1 & -2 & 0 & +\infty \\ +\infty & 1 & 2 & -2 & -1 & 0 & -\infty \end{pmatrix}.$$

With respect to section 6, we have changed ± 5 into $\pm\infty$, ± 3 into ± 2. The automorphism group acts by adding $m \in \mathbb{Z}_5$ everywhere.

The number of standard vertices is thus

$$N_{st}(\mathcal{D}) = 35.$$

19.3. **Edges of $\Gamma(\mathcal{D})$.** Consider the edges joining a vertex $X(0)$ ($X = S, A^+, \dots$) to other standard vertices in $\Gamma(\mathcal{D})$. This is determined by a pair of letters (α, β) ordered in the same way by π_t and π_b in $X(0)$.

If α, β are different from 0, the corresponding edge will join $X(0)$ to $Y(0)$ according to section 6.

The "new" edges correspond to $\beta = 0$, $\alpha = \pm 1, \pm 2$. One obtains for the other endpoint $Y(\alpha)$ of the corresponding edge:

X \ α	-2	-1	1	2
S	B^-	C^+	C^-	B^+
A^+	A^+	A^+	A^+	A^+
B^+	S	C^-	B^-	C^+
C^+	B^+	B^-	S	C^-

The other vertices are obtained from the involution.

19.4. **Default of vertices.** One has

$$\delta(S(m)) = 8, \quad \delta(A^{\pm}(m)) = 7, \quad \delta(B^{\pm}(m)) = \delta(C^{\pm}(m)) = 6.$$

19.5. **Default of the diagram.** The default of the diagram is

$$\delta(\mathcal{D}) = 115.$$

19.6. **Open linked vertices.** The open linked (to a standard vertex $X(0)$) vertices correspond to pairs of letters (α, β) which are **not** ordered in the same way by π_t and π_t. Therefore both α and β are different from 0, and the list is the same than in section 6.

20. The diagram $[9, 3](1)(1^3)$

20.1. **Alphabet, automorphism group, involutions.** We will use as alphabet

$$\mathcal{A} = \{\pm\infty, 0, a_1, a_2, b_1, b_2, c_1, c_2\}.$$

The automorphism group $\mathcal{G}$ has order 24. Every element of $\mathcal{G}$ fixes $0, -\infty, +\infty$ and preserves the partition of the remaining 6 letters into the three pairs $\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$. This property defines a subgroup $\mathcal{G}'$ of order 48 in the permutation group of these 6 letters. The group $\mathcal{G}'$ has a natural split homomorphism onto the permutation group of $\{a, b, c\}$, with section σ. The kernel of this homomorphism is isomorphic to $\{\pm 1\}^3$. Then $\mathcal{G}$ is the subgroup of index 2 of $\mathcal{G}'$ which is the kernel of the homomorphism $\mathcal{G}' \to \mathbb{Z}_2$ which sends $\sigma(\tau)$ to the signature of τ and a triple $(\varepsilon_a, \varepsilon_b, \varepsilon_c) \in \{\pm 1\}^3$ to $\varepsilon_a \varepsilon_b \varepsilon_c$[29].

There are three natural top/bottom exchanging involutions, denoted by I_a, I_b, I_c. The involution I_a fixes $0, a_1, a_2$ and exchanges $+\infty, -\infty$, b_1, b_2 and c_1, c_2.

20.2. **Superstandard vertices.** The superstandard vertices are the standard vertices which belong to the orbit under $\mathcal{G}$ of the vertex

$$S(b_1, a_1, c_1) := \begin{pmatrix} -\infty & b_2 & a_2 & b_1 & a_1 & c_1 & 0 & c_2 & \infty \\ \infty & b_1 & a_2 & b_2 & a_1 & c_2 & 0 & c_1 & -\infty \end{pmatrix}.$$

This vertex is fixed by I_a. In the graph $\Gamma(\mathcal{D})$, this vertex has valence 15. Standard vertices which are not superstandard have valence $11, 9, 8, 7$ or 6. In $\Gamma(\mathcal{D})$, the vertex $S(b_1, a_1, c_1)$ is connected to three other superstandard vertices

$$S(b_2, a_2, c_1) := \begin{pmatrix} -\infty & b_1 & a_1 & b_2 & a_2 & c_1 & 0 & c_2 & \infty \\ \infty & b_2 & a_1 & b_1 & a_2 & c_2 & 0 & c_1 & -\infty \end{pmatrix},$$

$$S(a_1, b_2, c_1) := \begin{pmatrix} -\infty & a_2 & b_1 & a_1 & b_2 & c_1 & 0 & c_2 & \infty \\ \infty & a_1 & b_1 & a_2 & b_2 & c_2 & 0 & c_1 & -\infty \end{pmatrix},$$

$$S(a_2, b_1, c_1) := \begin{pmatrix} -\infty & a_1 & b_2 & a_2 & b_1 & c_1 & 0 & c_2 & \infty \\ \infty & a_2 & b_2 & a_1 & b_1 & c_2 & 0 & c_1 & -\infty \end{pmatrix}.$$

These four vertices form a complete subgraph of $\Gamma(\mathcal{D})$ and is called a *cluster* of superstandard vertices. There are six such clusters. The cluster above is the c_1 cluster. The stabilizer in $\mathcal{G}$ of this cluster is the stabilizer of c_1 (or c_2), a cyclic subgroup of order 4.

---

[29] From the paper of C. Boissy (see footnote 7), the group $\mathcal{G}$ has order 24, and $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$ (seen as elements of $\mathcal{G}'$) are not in $\mathcal{G}$. Hence the homomorphism ϕ from $\mathcal{G}'$ to $\mathbb{Z}_2$ whose kernel is $\mathcal{G}$ sends a triple $(\varepsilon_a, \varepsilon_b, \varepsilon_c)$ to $\varepsilon_a \varepsilon_b \varepsilon_c$. To see that the composition $\phi \circ \sigma$ is not trivial (hence is the signature), we can consider the monotonous chain of length 4 starting from $S(b_1, a_1, c_1)$ given by the parameters (a_1, b_2).

The involution I_a fixes the c_1, c_2, b_1, b_2 clusters. It exchanges the a_1 and a_2 clusters. In the c_1 cluster, it fixes the vertices $S(b_1, a_1, c_1), S(b_2, a_2, c_1)$ and exchanges the other two vertices. On the other hand, the involution I_b exchanges $S(b_1, a_1, c_1), S(b_2, a_2, c_1)$ and fixes the other two vertices.

20.3. **Immediate neighborhood of a cluster.** The standard vertices which are connected (in $\Gamma(\mathcal{D})$) to a superstandard vertex in a cluster form the *immediate neighborhood* of this cluster. Besides the four vertices of the cluster, there are 48 such vertices (for each cluster), grouped into 4 groups of 12 because no vertex (except for the vertices of the cluster) is connected to two distinct superstandard vertices.

Consider the standard vertices connected to $S(b_1, a_1, c_1)$ which are not superstandard. There are 12 such vertices. Two of them have valence 11 and are exchanged by the involution I_a

$$A_t(b_1, a_1, c_1) := \begin{pmatrix} -\infty & a_2 & b_1 & a_1 & c_1 & 0 & c_2 & b_2 & +\infty \\ +\infty & a_1 & c_2 & b_1 & a_2 & b_2 & 0 & c_1 & -\infty \end{pmatrix},$$

$$A_b(b_1, a_1, c_1) := \begin{pmatrix} -\infty & a_1 & c_1 & b_2 & a_2 & b_1 & 0 & c_2 & +\infty \\ +\infty & a_2 & b_2 & a_1 & c_2 & 0 & c_1 & b_1 & -\infty \end{pmatrix}.$$

They correspond to the pairs $(b_1, c_1), (b_2, c_2)$ of letters for $S(b_1, a_1, c_1)$. The connexions of $S(b_1, a_1, c_1)$ to the other vertices of the cluster correspond to the pairs $(a_2, a_1), (b_1, a_1)$, (b_2, a_1).

Among the other vertices connected in $\Gamma(\mathcal{D})$ to $S(b_1, a_1, c_1)$, there are 5 of valence 9 and 5 of valence 7. Among each group of 5, there are two pairs whose elements are exchanged by the involution I_a and one element which is fixed by this involution.

The best way to organize the 12 non superstandard vertices connected to $\Gamma(\mathcal{D})$ is to observe that they may be grouped into three groups V_t, V_0, V_b with four elements each and the following properties:

- Two elements of the same group are connected by an edge in $\Gamma(\mathcal{D})$.
- Two element in distinct groups are **not** connected by an edge in $\Gamma(\mathcal{D})$.
- The involution I_a exchanges V_t and V_b, and fixes V_0.
- The vertex $A_t(b_1, a_1, c_1)$ belongs to V_t.
- The group V_t has, besides $A_t(b_1, a_1, c_1)$, two vertices of valence 7 and one of valence 9. Similarly for V_b.
- The group V_0 has three elements of valence 9 and one of valence 7.

20.4. **Notations for vertices in the immediate neighborhood.** The vertex of valence 9 in V_t is $B_t := B_t(b_1, a_1, c_1)$. One has

$$B_t := \begin{pmatrix} -\infty & b_1 & a_1 & c_1 & 0 & c_2 & b_2 & a_2 & +\infty \\ +\infty & b_2 & a_1 & c_2 & b_1 & a_2 & 0 & c_1 & -\infty \end{pmatrix},$$

$$B_b := \begin{pmatrix} -\infty & b_1 & a_1 & c_1 & b_2 & a_2 & 0 & c_2 & +\infty \\ +\infty & b_2 & a_1 & c_2 & 0 & c_1 & b_1 & a_2 & -\infty \end{pmatrix}.$$

They correspond to the pairs (a_2, c_1) and (a_2, c_2) of S.

The three vertices of valence 9 in V_0 are C_t, C_b, P (with P fixed by the involution I_a). They correspond to the pairs $(b_1, 0), (b_2, 0), (a_2, 0)$ of S. One has

$$C_t := \begin{pmatrix} -\infty & a_1 & c_1 & 0 & b_2 & a_2 & b_1 & c_2 & +\infty \\ +\infty & a_2 & b_2 & a_1 & c_2 & 0 & b_1 & c_1 & -\infty \end{pmatrix},$$

$$C_b := \begin{pmatrix} -\infty & a_2 & b_1 & a_1 & c_1 & 0 & b_2 & c_2 & +\infty \\ +\infty & a_1 & c_2 & 0 & b_1 & a_2 & b_2 & c_1 & -\infty \end{pmatrix},$$

$$P := \begin{pmatrix} -\infty & b_1 & a_1 & c_1 & 0 & b_2 & a_2 & c_2 & +\infty \\ +\infty & b_2 & a_1 & c_2 & 0 & b_1 & a_2 & c_1 & -\infty \end{pmatrix}.$$

The vertex in V_0 of valence 7, associated to the pair $(a_1, 0)$ of S, is

$$T := \begin{pmatrix} -\infty & c_1 & 0 & b_2 & a_2 & b_1 & a_1 & c_2 & +\infty \\ +\infty & c_2 & 0 & b_1 & a_2 & b_2 & a_1 & c_1 & -\infty \end{pmatrix}.$$

Finally, the two vertices of valence 7 in V_t are

$$E_t := \begin{pmatrix} -\infty & a_1 & c_1 & 0 & c_2 & b_2 & a_2 & b_1 & +\infty \\ +\infty & a_2 & b_2 & a_1 & c_2 & b_1 & 0 & c_1 & -\infty \end{pmatrix},$$

$$F_t := \begin{pmatrix} -\infty & c_1 & 0 & c_2 & b_2 & a_2 & b_1 & a_1 & +\infty \\ +\infty & c_2 & b_1 & a_2 & b_2 & a_1 & 0 & c_1 & -\infty \end{pmatrix}.$$

They are associated to the pairs $(b_1, c_2), (a_1, c_2)$ of S. The corresponding vertices in V_b, associated to the pairs $(b_2, c_1), (a_1, c_1)$ of S, are

$$E_b := \begin{pmatrix} -\infty & a_2 & b_1 & a_1 & c_1 & b_2 & 0 & c_2 & +\infty \\ +\infty & a_1 & c_2 & 0 & c_1 & b_1 & a_2 & b_2 & -\infty \end{pmatrix},$$

$$F_b := \begin{pmatrix} -\infty & c_1 & b_2 & a_2 & b_1 & a_1 & 0 & c_2 & +\infty \\ +\infty & c_2 & 0 & c_1 & b_1 & a_2 & b_2 & a_1 & -\infty \end{pmatrix}.$$

20.5. **Other edges in the immediate neighborhood of the cluster.** By "other edges" we mean edges whose endpoints belong to the immediate neighborhood of the cluster, are not superstandard, nor in the same group (V_t, V_0, V_b).

All three free edges from $T(b_1, a_1, c_1)$ are of this type, connecting this vertex with $C_t(a_1, b_2, c_1)$, $C_b(a_2, b_1, c_1)$ and $P(b_2, a_2, c_1)$.

All three free edges from $F_t(b_1, a_1, c_1)$ are also of this type, connecting this vertex with $E_t(a_1, b_2, c_1)$, $A_t(a_2, b_1, c_1)$ and $B_t(b_2, a_2, c_1)$.

The other edges of this form are obtained by the automorphisms and involutions of the diagram.

This leaves the T and F vertices with no free edges, the E vertices with 2 free edges, the B, C, P vertices with 4 free edges and the A vertices with 6 free edges.

20.6. **Edges with endpoints in immediate neighborhood of distinct clusters.** There is an edge between $A_t(b_1, a_1, c_1)$ and the vertex $C_b(c_1, a_2, b_1)$ in the immediate neighborhood of the b_1 cluster.

The involution I_a takes this edge to an edge between $A_b(b_1, a_1, c_1)$ and $C_t(c_1, a_2, b_1)$. Notice that $S(c_1, a_2, b_1)$ is fixed by I_a.

There is an edge between $B_t(b_1, a_1, c_1)$ and the vertex $P(c_1, b_1, a_1)$ in the immediate neighborhood of the a_1 cluster.

The involution I_a takes this edge to an edge between $B_b(b_1, a_1, c_1)$ and $P(c_1, b_2, a_2)$, in the immediate neighborhood of the a_2 cluster. Recall that the involution I_a exchanges the a_1 and the a_2 clusters.

All edges of this type are deduced from theses four edges by the automorphism group G.

After taking these edges into account, the P and E vertices are left with 2 free edges, the C_t, C_b and B vertices with 3 free edges and the A vertices with 5 free edges. The free endpoints of these free edges do not belong to the immediate neighborhood of any cluster.

We observe that the different type of standard vertices encountered so far can be recognized by the position of 0 in the top and bottom line:

$$\begin{array}{lllll} S \to (7,7) & R \to (6,6) & P \to (5,5) & Q \to (4,4) & T \to (3,3) \\ A_t \to (6,7) & A_b \to (7,6) & B_t \to (5,7) & B_b \to (7,5) & \\ C_t \to (4,6) & C_b \to (6,4) & E_t \to (4,7) & E_b \to (7,4) & \\ F_t \to (3,7) & F_b \to (7,3) & G_t \to (3,5) & G_b \to (5,3) & \\ H_t \to (3,4) & H_b \to (4,3) & I_t \to (3,6) & I_b \to (6,3) & \end{array}$$

20.7. **Vertices connected to the immediate neighborhood of clusters.** We start with the two free edges from $P(b_1, a_1, c_1)$, which are symmetric w.r.t. the involution I_a. Denote their free endpoints by

$$G_t := \begin{pmatrix} -\infty & c_1 & 0 & b_2 & a_2 & c_2 & b_1 & a_1 & +\infty \\ +\infty & c_2 & b_2 & a_1 & 0 & b_1 & a_2 & c_1 & -\infty \end{pmatrix},$$

$$G_b := \begin{pmatrix} -\infty & c_1 & b_1 & a_1 & 0 & b_2 & a_2 & c_2 & +\infty \\ +\infty & c_2 & 0 & b_1 & a_2 & c_1 & b_2 & a_1 & -\infty \end{pmatrix}.$$

Notice that we have $G_t \to (3,5)$ and $G_b \to (5,3)$. Both vertices have valence 7. The vertex $G_t(b_1, a_1, c_1)$ is also connected to $A_b(c_1, b_2, a_2)$, $B_t(b_2, a_2, c_1)$ and $B_b(a_2, c_2, b_1)$. The vertex $G_b(b_1, a_1, c_1)$ is also connected to $A_t(c_1, b_1, a_1)$, $B_b(b_2, a_2, c_1)$ and $B_t(a_1, c_1, b_1)$.

Now the B vertices have one free edge, the E vertices have 2, the C and G have 3 and the A have 4.

We look at the free endpoints of the remaining free edges from the B vertices, which are denoted by H_b for B_t and H_t for B_b

$$H_t := \begin{pmatrix} -\infty & a_2 & 0 & b_1 & a_1 & c_1 & b_2 & c_2 & +\infty \\ +\infty & a_1 & c_2 & 0 & b_2 & c_1 & b_1 & a_2 & -\infty \end{pmatrix},$$

$$H_b := \begin{pmatrix} -\infty & a_1 & c_1 & 0 & b_1 & c_2 & b_2 & a_2 & +\infty \\ +\infty & a_2 & 0 & b_2 & a_1 & c_2 & b_1 & c_1 & -\infty \end{pmatrix}.$$

Notice that we have $H_t \to (3,4)$ and $H_b \to (4,3)$. Both vertices have valence 6. Besides B_b, the vertex $H_t(b_1, a_1, c_1)$ is also connected to $E_t(c_2, b_1, a_2)$, $C_b(b_2, c_2, a_2)$, $G_b(b_2, a_2, c_1)$, $H_b(b_1, c_1, a_2)$. Besides $B_t(b_1, a_1, c_1)$, the vertex $H_b(b_1, a_1, c_1)$ is also connected to $E_b(c_2, b_2, a_1)$, $C_t(b_2, c_1, a_1)$, $G_t(b_2, a_2, c_1)$, $H_t(b_1, c_2, a_1)$.

Now the E vertices have only 1 free edge (of valence 8), the C and G vertices have 2 free edges (one of valence 6, one of valence 8), the H vertices have only 1 free edge (of valence 8) and the A vertices have 4 free edges (2 of valence 6, 2 of valence 8). We have make an "abus de langage" by writing the valence of a free edge instead of its free endpoint.

We look at the free endpoints of the remaining free edges of valence 6 from the C_t and C_b vertices, which are denoted by I_t and I_b respectively

$$I_t := \begin{pmatrix} -\infty & c_1 & 0 & b_2 & a_2 & b_1 & c_2 & a_1 & +\infty \\ +\infty & c_2 & a_2 & b_2 & a_1 & 0 & b_1 & c_1 & -\infty \end{pmatrix},$$

$$I_b := \begin{pmatrix} -\infty & c_1 & a_2 & b_1 & a_1 & 0 & b_2 & c_2 & +\infty \\ +\infty & c_2 & 0 & b_1 & a_2 & b_2 & c_1 & a_1 & -\infty \end{pmatrix}.$$

Notice that we have $I_t \to (3,6)$ and $I_b \to (6,3)$. Both vertices have valence 6. Besides C_t, the vertex $I_t(b_1,a_1,c_1)$ is also connected to $A_t(a_2,b_1,c_1)$, $A_b(c_1,a_2,b_1)$ and $G_b(b_1,c_1,a_2)$. Besides C_b, the vertex $I_b(b_1,a_1,c_1)$ is also connected to $A_b(a_1,b_2,c_1)$, $A_t(c_1,a_2,b_1)$ and $G_t(b_1,c_2,a_1)$.

Now the E, C, G, H vertices have only one free edge (of valence 8) while the A and I vertices have 2 free edges (both of valence 8).

The two vertices of valence 8 connected to A_t correspond to positions $(4,4)$ and $(6,6)$ for the 0 letter. We denote them by Q and R respectively.

$$Q := \begin{pmatrix} -\infty & a_1 & c_1 & 0 & a_2 & b_1 & c_2 & b_2 & +\infty \\ +\infty & a_2 & b_2 & 0 & a_1 & c_2 & b_1 & c_1 & -\infty \end{pmatrix},$$

$$R := \begin{pmatrix} -\infty & c_1 & a_2 & b_1 & a_1 & 0 & c_2 & b_2 & +\infty \\ +\infty & c_2 & b_1 & a_2 & b_2 & 0 & c_1 & a_1 & -\infty \end{pmatrix}.$$

Besides $A_t(b_1,a_1,c_1)$, $Q(b_1,a_1,c_1)$ is connected to $A_b(c_1,a_2,b_1)$, $E_t(b_2,c_1,a_1)$, $E_b(c_2,b_2,a_1)$, $I_t(a_1,b_2,c_1)$, $I_b(a_1,c_1,b_1)$, $G_t(a_1,b_2,c_1)$, $G_b(a_1,c_1,b_1)$.

Besides $A_t(b_1,a_1,c_1)$, the vertex $R(b_1,a_1,c_1)$ is connected to $A_b(a_1,b_2,c_1)$, $C_t(c_1,b_1,a_1)$, $C_b(c_1,a_2,b_1)$, $I_t(c_1,b_1,a_1)$, $I_b(c_1,a_2,b_1)$, $H_t(c_1,a_2,b_1)$, $H_b(c_1,b_1,a_1)$.

There are no more free edges so that one can hope that we have now all the standard vertices.

20.8. **Number of standard vertices.** There are 21 orbits of standard vertices for the action of G: $Q,R,S,T,P,A_t,A_b,B_t,B_b,C_t,C_b,E_t,E_b,F_t,F_b,G_t,G_b,H_t,H_b,I_t,I_b$. Therefore the total number of standard vertices is

$$N_{st} = 24 \times 21 = 504.$$

The default of $\Gamma(\mathcal{D})$ is equal to $2052 = 12 \times 171$.

20.9. **Up to height** 4**.** There are $3528 = 7 \times 21 \times 24$ vertices with $H_t = 2, H_b = 4$ and 3528 vertices with $H_b = 2, H_t = 4$.

Therefore there are 504 cycles of height 3, each type and each length $\ell \in \{1,2,3,4,5,6,7\}$.

Therefore there are $10584 = 21 \times 21 \times 24$ vertices with $H_t = H = 4$, and 10584 with $H_b = H = 4$.

The number of vertices with $H_b = H_t = 4$ is $4104 = 24 \times 171$. Therefore there are $6480 = 24 \times 270$ vertices with $H_t = 4, H_b = 6$ and 6480 with $H_b = 4, H_t = 6$.

20.10. **Cycles of height** 5**.** There are $2016 = 24 \times 84$ cycles of top type,height 5 and length 1.

There are $1176 = 24 \times 49$ cycles of top type,height 5 and length 2. Among these, $336 = 14 \times 24$ have two vertices of height 4, and $840 = 24 \times 35$ have one vertex of height 4, one vertex of height 6.

There are $576 = 24 \times 24$ cycles of top type,height 5 and length 3. Among these, $96 = 4 \times 24$ have three vertices of height 4, $240 = 24 \times 10$ have two vertices of height 4, one vertex of height 6, and $240 = 24 \times 10$ have one vertex of height 4, two vertices of height 6.

There are $360 = 24 \times 15$ cycles of top type,height 5 and length 4. Among these, $72 = 3 \times 24$ have four vertices of height 4, $72 = 24 \times 3$ have three vertices of height 4, one vertex of height 6, $144 = 24 \times 6$ have two vertices of height 4, two vertices of height 6, and $72 = 24 \times 3$ have three vertices of height 6, one vertex of height 4.

There are $240 = 24 \times 10$ cycles of top type,height 5 and length 5. Among these, 48 have five vertices of height 4, 48 have four vertices of height 4, one vertex of height 6, 48 have three vertices of height 4, two vertices of height 6, 48 have two vertices of height 4, three vertices of height 6, and 48 have one vertex of height 4, four vertices of height 6.

There are $120 = 24 \times 5$ cycles of top type,height 5 and length 6. For each $j \in \{1, 2, 3, 4, 5\}$, there are 24 such cycles which contain j vertices of height 4 and $6 - j$ vertices of height 6.

Summing up, there are $2976 = 124 \times 24$ vertices with $H_t = H = 6$, and 2976 with $H_b = H = 6$.

20.11. **Vertices of height 6.** We consider the vertices V with $H_t = H = 6$. We call $\mathcal{C}_t$ the pure cycle of top type , height 5 through V, and $\mathcal{C}_b$ the pure cycle of bottom type through V. The height of $\mathcal{C}_b$ is equal to 5 or 7, corresponding for V to $H_b = 6$ or 8. We call ℓ_t, ℓ_b the lengths of $\mathcal{C}_t$, $\mathcal{C}_b$ respectively.

Among the $840 = 24 \times 35$ vertices V with $\ell_t = 2$, $288 = 24 \times 12$ have $H_b = 6$ (they are then the middle vertices of monotonous chains of length 6) and $552 = 24 \times 23$ have $H_b = 8$: $360 = 24 \times 15$ with $\ell_b = 1$, $120 = 24 \times 5$ with $\ell_b = 2$, 24 with $\ell_b = 3$, 24 with $\ell_b = 4$, 24 with $\ell_b = 5$.

Among the $720 = 24 \times 30$ vertices V with $\ell_t = 3$,

- $240 = 24 \times 10$ are linked through $\mathcal{C}_t$ to two vertices of height 4 (A_t/E_t, A_b/E_t, Q/F_b, R/F_t, T/G_t, T/G_b, F_t/G_t, F_b, G_b, H_t/I_t, H_b/I_b)
- $480 = 24 \times 20$ come in pairs linked through $\mathcal{C}_t$ to a single vertex of height 4 (B_t, B_b, C_t, C_b, E_b, E_b, H_t, H_b, I_t, I_b).

Among these 720 vertices,

- $288 = 24 \times 12$ have $H_b = 6$,
- $312 = 24 \times 13$ have $H_b = 8$, $\ell_b = 1$,
- 48 have $H_b = 8$, $\ell_b = 2$,
- 24 have $H_b = 8$, $\ell_b = 3$,
- 24 have $H_b = 8$, $\ell_b = 4$,
- 24 have $H_b = 8$, $\ell_b = 5$.

Among the $576 = 24 \times 24$ vertices V with $\ell_t = 4$,

- $72 = 24 \times 3$ are linked through $\mathcal{C}_t$ to three vertices of height 4 ($B_t/B_b/I_t$, $C_t/C_b/H_b$, $Q/R/E_t$). They all have $H_b = 8$, $\ell_b = 1$.
- $288 = 24 \times 12$ vertices come in pairs linked through $\mathcal{C}_t$ to two vertices of height 4 (P/F_b, P/F_t, T/I_t, T/H_b, F_t/H_t, F_b/I_b); $120 = 24 \times 5$ have $H_b = 6$, $96 = 24 \times 4$ have $H_b = 8, \ell_b = 1$, $48 = 24 \times 2$ have $H_b = 8, \ell_b = 2$, 24 have $H_b = 8, \ell_b = 3$
- $216 = 24 \times 9$ vertices come in triples joined through $\mathcal{C}_t$ to a single vertex of height 4 (E_t, G_t, G_b); $96 = 24 \times 4$ have $H_b = 6$, $72 = 24 \times 3$ have $H_b = 8, \ell_b = 1$, 24 have $H_b = 8, \ell_b = 3$, 24 have $H_b = 8, \ell_b = 5$.

Among the $480 = 24 \times 20$ vertices V with $\ell_t = 5$,

- 48 are linked through $\mathcal{C}_t$ to four vertices of height 4 ($A_t/C_t/E_b/G_t$, $A_b/B_t/E_b/G_b$); these vertices have $H_b = 8, \ell_b = 1$.
- $96 = 24\times4$ come in pairs linked through $\mathcal{C}_t$ to three vertices of height 4 ($B_b/E_t/H_b$, $C_b/E_t/I_t$); 48 have $H_b = 8, \ell_b = 1$, 48 have $H_b = 8, \ell_b = 2$.
- $144 = 24 \times 6$ come in triples linked through $\mathcal{C}_t$ to two vertices of height 4 (Q/F_t, R/F_b); 48 have $H_b = 6$, 48 have $H_b = 8, \ell_b = 1$, 24 have $H_b = 8, \ell_b = 2$, 24 have $H_b = 8, \ell_b = 4$.
- $192 = 24 \times 8$ come in quadruples linked through $\mathcal{C}_t$ to a single vertex of height 4 (H_t, I_b); 48 have $H_b = 6$, 48 have $H_b = 8, \ell_b = 1$, 48 have $H_b = 8, \ell_b = 2$, 24 have $H_b = 8, \ell_b = 4$, 24 have $H_b = 8, \ell_b = 5$.

Among the $360 = 24 \times 15$ vertices V with $\ell_t = 6$,

- 24 are linked through $\mathcal{C}_t$ to five vertices of height 4 ($P/Q/R/F_t/F_b$); they have $H_b = 8, \ell_b = 1$.
- 48 come in pairs linked through $\mathcal{C}_t$ to four vertices of height 4 ($B_t/C_t/H_t/I_b$); 24 have $H_b = 8, \ell_b = 1$, 24 have $H_b = 8, \ell_b = 2$.
- $72 = 24 \times 3$ come in triples linked through $\mathcal{C}_t$ to three vertices of height 4 ($E_t/G_t/G_b$); 24 have $H_b = 8, \ell_b = 1$, 24 have $H_b = 8, \ell_b = 2$, 24 have $H_b = 8, \ell_b = 3$.
- $96 = 24 \times 4$ come in quadruples linked through $\mathcal{C}_t$ to two vertices of height 4 (H_b/I_t); 24 have $H_b = 8, \ell_b = 1$, 24 have $H_b = 8, \ell_b = 2$, 24 have $H_b = 8, \ell_b = 3$, 24 have $H_b = 8, \ell_b = 4$.
- $120 = 24 \times 5$ come in quintuples linked through $\mathcal{C}_t$ to a single vertex of height 4 (T); 24 have $H_b = 8, \ell_b = 1$, 24 have $H_b = 8, \ell_b = 2$, 24 have $H_b = 8, \ell_b = 3$, 24 have $H_b = 8, \ell_b = 4$, 24 have $H_b = 8, \ell_b = 5$.

Summing up, there are $888 = 24 \times 37$ vertices with $H_t = H_b = 6$, $2088 = 24 \times 87$ vertices with $H_t = 6, H_b = 8$, and 2088 vertices with $H_t = 8, H_b = 6$. Of the 2088 vertices with $H_t = 6, H_b = 8$, $1224 = 24 \times 51$ have $\ell_b = 1$ and $864 = 24 \times 36$ have $\ell_b > 1$: $432 = 24 \times 18$, with $\ell_b = 2$, $168 = 24 \times 7$ with $\ell_b = 3$, $144 = 24 \times 6$ with $\ell_b = 4$ and $120 = 24 \times 5$ with $\ell_b = 5$.

20.12. **Cycles of height 7.** We have seen that there are $1224 = 24 \times 51$ cycles of each type, height 7, length 1.

There are $264 = 24 \times 11$ pure cycles of bottom type, height 7, length 2. $168 = 24 \times 7$ are the middle cycles of a monotonous chain of length 7. The other $96 = 24 \times 4$ have one vertex of height 6 and one vertex with $H_b = 8, H_t = 10$. The top cycles through these vertices have length 1.

There are $72 = 24 \times 3$ pure cycles of bottom type, height 7, length 3. 24 of these cycles contain only vertices of height 6. The other 48 contain 2 vertices of height 6 and one vertex with $H_b = 8, H_t = 10$. The top cycles through these vertices have length 1.

There are $48 = 24 \times 3$ pure cycles of bottom type, height 7, length 4. 24 of these cycles contain only vertices of height 6. The other 24 contain 2 vertices of height 6 and two vertices with $H_b = 8, H_t = 10$, giving 48 such vertices. The top cycle through these vertices has length 1 for 24 of them, length 2 for the other 24. The other vertex of these pure cycles of top type, height 9, length 2 has $H_t = 10, H_b = 12$; the bottom cycle through it (of height 11) has length 1.

There are 24 pure cycles of bottom type, height 7, length 5. Their vertices have all height 6.

21. THE DIAGRAM $[7,3](2,2)Odd$

21.1. **Alphabet, Automorphisms, Involutions.** We choose as alphabet $\mathcal{A} = \{\pm\infty, a^+, a^-\} \cup \mathbb{Z}_3$. The automorphism group is isomorphic to $\mathbb{Z}_3$, acting by addition on $\mathbb{Z}_3$ and fixing the other elements of $\mathcal{A}$. There are three time-reversing involutions, indexed by $\mathbb{Z}_3$. The involution I_0 fixes 0 and exchanges $\pm\infty$, ± 1, $a^{\pm}$.

21.2. **Standard vertices.** There are 33 standard vertices. They are indexed by a letter and an element of $\mathbb{Z}_3$. The automorphism group acts by $j.X(i) = X(i+j)$. When $i = 0$, the 11 vertices are

$$S := \begin{pmatrix} -\infty & 1 & -1 & 0 & a^- & a^+ & +\infty \\ +\infty & -1 & 1 & 0 & a^+ & a^- & -\infty \end{pmatrix},$$

$$P := \begin{pmatrix} -\infty & a^- & -1 & a^+ & 0 & 1 & +\infty \\ +\infty & a^+ & 1 & a^- & 0 & -1 & -\infty \end{pmatrix},$$

$$Q := \begin{pmatrix} -\infty & a^- & 0 & 1 & a^+ & -1 & +\infty \\ +\infty & a^+ & 0 & -1 & a^- & 1 & -\infty \end{pmatrix},$$

$$A^+ := \begin{pmatrix} -\infty & a^- & a^+ & 1 & -1 & 0 & +\infty \\ +\infty & a^+ & -1 & 1 & 0 & a^- & -\infty \end{pmatrix},$$

$$A^- := \begin{pmatrix} -\infty & a^- & 1 & -1 & 0 & a^+ & +\infty \\ +\infty & a^+ & a^- & -1 & 1 & 0 & -\infty \end{pmatrix},$$

$$B^+ := \begin{pmatrix} -\infty & -1 & 0 & a^- & a^+ & 1 & +\infty \\ +\infty & 0 & a^+ & -1 & 1 & a^- & -\infty \end{pmatrix},$$

$$B^- := \begin{pmatrix} -\infty & 0 & a^- & 1 & -1 & a^+ & +\infty \\ +\infty & 1 & 0 & a^+ & a^- & -1 & -\infty \end{pmatrix},$$

$$C^+ := \begin{pmatrix} -\infty & -1 & 0 & a^- & 1 & a^+ & +\infty \\ +\infty & 0 & a^+ & a^- & -1 & 1 & -\infty \end{pmatrix},$$

$$C^- := \begin{pmatrix} -\infty & 0 & a^- & a^+ & 1 & -1 & +\infty \\ +\infty & 1 & 0 & a^+ & -1 & a^- & -\infty \end{pmatrix},$$

$$D^+ := \begin{pmatrix} -\infty & 0 & a^- & -1 & a^+ & 1 & +\infty \\ +\infty & 1 & a^- & 0 & a^+ & -1 & -\infty \end{pmatrix},$$

$$D^- := \begin{pmatrix} -\infty & -1 & a^+ & 0 & a^- & 1 & +\infty \\ +\infty & 0 & a^+ & 1 & a^- & -1 & -\infty \end{pmatrix}.$$

The involution I_0 fixes P, Q, S and exchanges $A^{\pm}, B^{\pm}, C^{\pm}, D^{\pm}$.

21.3. **The graph $\Gamma(\mathcal{D})$.**

- S has valence 8 and is connected to $S(\pm 1), A^{\pm}, B^{\pm}, C^{\pm}$;
- P has valence 4 and is connected to $D^{\pm}, C^+(1), C^-(-1)$;
- Q has valence 4 and is connected to $B^+(1), B^-(-1), D^+(1), D^-(-1)$;
- A^+ has valence 5 and is connected to $S, B^+, B^+(-1), C^-, C^-(1)$;
- B^+ has valence 6 and is connected to $S, Q(-1), A^+, A^+(1), C^-, D^+$;
- C^+ has valence 5 and is connected to $S, P(-1), A^-, A^-(1), B^-$;
- D^+ has valence 4 and is connected to $P, Q(-1), B^+, D^-$.

The default of the diagram is $3 \times 28 = 84$.

21.4. **Up to height 4.** There are $33 = 3 \times 11$ cycles of height 1 of top type (resp. bottom type). There are 165 vertices with $H_t = 2, H_b = 4$ and 165 with $H_b = 2, H_t = 4$. Therefore there are 165 cycles of top type (resp. bottom type) and height 3, 33 for each length $1, 2, 3, 4, 5$. Therefore there are 330 vertices with $H_t(\pi) = H(\pi) = 4$, and 330 with $H_b(\pi) = H(\pi) = 4$. To each edge in $\Gamma(\mathcal{D})$ are associated two vertices with $H_t = H_b = 4$. Therefore there are 168 such vertices, leaving 162 vertices with $H_t = 4, H_b = 6$ and 162 with $H_b = 4, H_t = 6$.

21.5. **Cycles of height 5 and vertices of height 6.** There are $72 = 3 \times 24$ pure cycles of top type, height 5, length 1.

There are $24 = 3 \times 8$ pure cycles of top type, height 5, length 2. 15 of these have two vertices of height 4, joining $P(i)$ to $A^-(i-1)$, $P(i)$ to $Q(i+1)$, $Q(i)$ to $A^+(i+1)$, $B^+(i)$ to $C^-(i-1)$, $B^-(i)$ to $C^-(i)$. 9 of these have one vertex of height 4 and one vertex of height 6.

There are $12 = 3 \times 4$ pure cycles of top type, height 5, length 3. Six of these have two vertices of height 4 (linked to $P(i), A^+(i-1)$, resp. to $Q(i), A^-(i+1)$) and one of height 6. The other six have three vertices of height 4 (joining $B^+(i), C^+(i), D^-(i)$, resp. $C^+(i), D^+(i-1), B^-(i-1)$).

There are $6 = 3 \times 2$ pure cycles of top type, height 5, length 4. Three of these have three vertices of height 4 (linked to $C^-(i), D^+(i), D^-(i+1)$) and one of height 6. The other three have four vertices of height 4, linked to $P(i), Q(i-1), A^+(i+1), A^-(i+1)$.

Altogether, there are $18 = 3 \times 6$ vertices with $H_t = H = 6$.

There are 27 vertices of height 6: 9 with $H_t = H_b = 6$, 9 with $H_t = 8, H_b = 6$, 9 with $H_t = 6, H_b = 8$. The pure cycles of top type through the vertices with $H_t = 8$ have length 1. Similarly for the vertices with $H_b = 8$. Finally, the vertices with $H_t = H_b = 6$ are the middle vertices of monotonous chains of length 6: Three joining $C^+(i)$ to $C^-(I)$, which are preserved by the involution; three joining $D^+(i)$ to $A^+(i)/Q(i+1)$, and three joining $D^-(i)$ to $A^-(i)/Q(i-1)$.

21.6. **Summary.** There are

- 33 vertices of height 0;
- 330 vertices of height 2;
- 492 vertices of height 4;
- 27 vertices of height 6;

There are apparently 882 vertices in the diagram.

22. THE DIAGRAM $[7,3](1)(3)$

22.1. **Alphabet, Automorphisms, Involutions.** We choose as alphabet

$$\mathcal{A} := \{\pm\infty, 0, a_0, a_1, b_0, b_1\}.$$

The automorphism group has order 2, the non trivial element exchanges a_0/a_1, $b_0/b_1$and fixes the other letters.

There are two involutions I_0 and I_1. The involution I_0 exchanges $+\infty/-\infty$, a_0/b_0, a_1/b_1. The involution I_1 exchanges $+\infty/-\infty$, a_0/b_1, b_0/a_1.

22.2. **Standard vertices.** There are 16 standard vertices which are denoted by $S(i), T(i)$, $A^{\pm}(i), B^{\pm}(i), C^{\pm}(i)$, $i \in \mathbb{Z}_2$.

The nontrivial automorphism acts by $X(i) \to X(i+1)$. The involution I_0 fixes $S(0)$ and $S(1)$, exchanges $T(0)/T(1)$ and also exhanges $X^+(i)/X^-(i)$, for $X = A, B, C, D$ and $i \in \mathbb{Z}_2$.

One has

$$S(0) := \begin{pmatrix} -\infty & b_1 & a_1 & a_0 & 0 & b_0 & +\infty \\ +\infty & a_1 & b_1 & b_0 & 0 & a_0 & -\infty \end{pmatrix},$$

$$T(0) := \begin{pmatrix} -\infty & a_0 & 0 & a_1 & b_0 & b_1 & +\infty \\ +\infty & b_1 & 0 & b_0 & a_1 & a_0 & -\infty \end{pmatrix},$$

$$A^+(0) := \begin{pmatrix} -\infty & a_1 & a_0 & 0 & b_0 & b_1 & +\infty \\ +\infty & b_0 & a_1 & b_1 & 0 & a_0 & -\infty \end{pmatrix},$$

$$B^+(0) := \begin{pmatrix} -\infty & a_0 & 0 & b_0 & b_1 & a_1 & +\infty \\ +\infty & b_1 & b_0 & a_1 & 0 & a_0 & -\infty \end{pmatrix},$$

$$C^+(0) := \begin{pmatrix} -\infty & a_0 & a_1 & 0 & b_0 & b_1 & +\infty \\ +\infty & b_1 & 0 & a_0 & b_0 & a_1 & -\infty \end{pmatrix},$$

$$A^-(0) := \begin{pmatrix} -\infty & a_0 & b_1 & a_1 & 0 & b_0 & +\infty \\ +\infty & b_1 & b_0 & 0 & a_0 & a_1 & -\infty \end{pmatrix},$$

$$B^-(0) := \begin{pmatrix} -\infty & a_1 & a_0 & b_1 & 0 & b_0 & +\infty \\ +\infty & b_0 & 0 & a_0 & a_1 & b_1 & -\infty \end{pmatrix},$$

$$C^-(0) := \begin{pmatrix} -\infty & a_1 & 0 & b_0 & a_0 & b_1 & +\infty \\ +\infty & b_0 & b_1 & 0 & a_0 & a_1 & -\infty \end{pmatrix}.$$

22.3. **The graph $\Gamma(\mathcal{D})$.**

- The vertex $S(0)$ has valence 6; it is connected to $A^{\pm}(0), B^{\pm}(0), C^{\pm}(1)$;
- The vertex $T(0)$ has valence 2; it is connected to $A^+(0), A^-(1)$;
- The vertex $A^+(0)$ has valence 4; it is connected to $S(0), T(0), B^+(0), C^+(0)$;
- The vertex $A^-(0)$ has valence 4; it is connected to $S(0), T(1), B^-(0), C^-(0)$;
- The vertex $B^+(0)$ has valence 2; it is connected to $S, A^+(0)$;
- The vertex $B^-(0)$ has valence 2; it is connected to $S, A^-(0)$;
- The vertex $C^+(0)$ has valence 3; it is connected to $S(1), A^+(0), C^-(0)$;
- The vertex $C^-(0)$ has valence 3; it is connected to $S(1), A^-(0), C^+(0)$.

The default of the diagram is equal to 26.

22.4. **Up to height** 4**.** There are 16 cycles of height 1 and top (resp. bottom type). There are 80 vertices with $H_t = 2, H_b = 4$ and 80 with $H_b = 2, H_t = 4$. Therefore there are 80 cycles of top type (resp. bottom type) and height 3, 16 for each length $1, 2, 3, 4, 5$. Therefore there are 160 vertices with $H_t(\pi) = H(\pi) = 4$, and 160 with $H_b(\pi) = H(\pi) = 4$. To each edge in $\Gamma(\mathcal{D})$ are associated two vertices with $H_t = H_b = 4$. Therefore there are 52 such vertices, leaving 108 vertices with $H_t = 4, H_b = 6$ and 108 with $H_b = 4, H_t = 6$.

22.5. **Cycles of height** 5 **and vertices of height** 6**.** There are $44 = 2 \times 22$ pure cycles of top type, height 5, length 1.

There are $22 = 2 \times 11$ pure cycles of top type, height 5, length 2. 6 of these have two vertices of height 4, joining $T(i)$ to $C^-(i-1)$, $B^+(i)$ to $C^-(i-1)$, $A^-(i)$ to $C^-(i+1)$. 16 of these have one vertex of height 4 and one vertex of height 6 (from $S(i), T(i), A^-(i), B^-(i)C^+(i), B^-(i), A^+(i), B^+(i)$).

There are $12 = 2 \times 6$ pure cycles of top type, height 5, length 3. Four of these have three vertices of height 4, joining $T(i)/B^-(i-1)/C^+(i)$ and $A^+(i)/A^-(i-1)/C^+(i-1)$. Four of them have two vertices of height 4 ($T(i)/B^+(i)$ and $B^+(i)/A^-(i)$) and a vertex of height 6. Four of them have one vertex of height 4 ($C^-(i)$ and $B^-(i)$) and two vertices of height 6.

There are $6 = 2 \times 3$ pure cycles of top type, height 5, length 4. Two of these have one vertex of height 6 and three of height 4 ($A^+(i)/B^-(i)/C^-(i)$), another two have two vertices of height 6 and two of height 4 ($B^+(i)/C^+(i)$), and the last two have three vertices of height 6 and one of height 4 ($T(i)$).

Altogether, there are $42 = 2 \times 21$ vertices with $H_t = H = 6$.

There are 80 vertices of height 6: 6 with $H_t = H_b = 6$,36 with $H_t = 8, H_b = 6$, 36 with $H_t = 6, H_b = 8$. The vertices with $H_t = H_b = 6$ are middle vertices of monotonous chains joining $C^+(i)/B^+(i-1)$, $B^-(i)/C^-(i-1)$, $B^-(i)/B^+(i)$. Of the 36 vertices with $H_t = 6, H_b = 8$, 24 have a bottom cycle of height 7, length 1 through them. There are 4 cycles of height 7, bottom type, length 2. Two are the middle elements of monotonous chains of length 7 joining $A^+(i-1)$ to $B^+(i)/C^+(i)$. The other two, linked to $T(i)$, have an element of height 6 and an element with $H_b = 8, H_t = 10$. There are also two cycles of height 7, bottom type, length 3. They join $T(i), B^+(i), C^-(i-1)$. There are 4 vertices of height 8, 2 with $H_b = 8, H_t = 10$ and two with $H_t = 8, H_b = 10$. The top cycles through the vertices with $H_b = 8, H_t = 10$ have length 1.

22.6. **Summary.** There are

- 16 vertices of height 0;
- 160 vertices of height 2;
- 268 vertices of height 4;
- 80 vertices of height 6;
- 4 vertices of height 8.

There are apparently 528 vertices in the diagram.

23. THE DIAGRAM $[8,4](6)E$

23.1. **Alphabet, Automorphisms, Involutions.** We use the alphabet $\mathcal{A} = \{\pm\infty, \pm 1, \pm 2, \pm 3\}$. There is no non trivial automorphism. The involution exchanges $\pm\infty, \pm 1, \pm 2, \pm 3$.

23.2. **Standard vertices.** There are 44 standard vertices. Two of them are fixed by the involution

$$X := \begin{pmatrix} -\infty & 2 & -2 & 1 & 3 & -3 & -1 & +\infty \\ +\infty & -2 & 2 & -1 & -3 & 3 & 1 & -\infty \end{pmatrix},$$

$$Y := \begin{pmatrix} -\infty & 2 & -1 & 1 & -2 & -3 & 3 & +\infty \\ +\infty & -2 & 1 & -1 & 2 & 3 & -3 & -\infty \end{pmatrix},$$

The other 42 come in pairs of vertices exchanged by the involution.

$$A^+ := \begin{pmatrix} -\infty & -1 & 1 & -2 & 3 & -3 & 2 & +\infty \\ +\infty & 3 & 1 & -1 & 2 & -3 & -2 & -\infty \end{pmatrix},$$

$$A^- := \begin{pmatrix} -\infty & -3 & -1 & 1 & -2 & 3 & 2 & +\infty \\ +\infty & 1 & -1 & 2 & -3 & 3 & -2 & -\infty \end{pmatrix},$$

$$B^+ := \begin{pmatrix} -\infty & -3 & 2 & -1 & 1 & -2 & 3 & +\infty \\ +\infty & 1 & -1 & 2 & 3 & -3 & -2 & -\infty \end{pmatrix},$$

$$B^- := \begin{pmatrix} -\infty & -1 & 1 & -2 & -3 & 3 & 2 & +\infty \\ +\infty & 3 & -2 & 1 & -1 & 2 & -3 & -\infty \end{pmatrix},$$

$$C^+ := \begin{pmatrix} -\infty & -2 & -3 & 2 & -1 & 1 & 3 & +\infty \\ +\infty & -1 & 2 & 3 & -3 & -2 & 1 & -\infty \end{pmatrix},$$

$$C^- := \begin{pmatrix} -\infty & 1 & -2 & -3 & 3 & 2 & -1 & +\infty \\ +\infty & 2 & 3 & -2 & 1 & -1 & -3 & -\infty \end{pmatrix},$$

$$D^+ := \begin{pmatrix} -\infty & 1 & -2 & -3 & 2 & -1 & 3 & +\infty \\ +\infty & 2 & 3 & -3 & -2 & 1 & -1 & -\infty \end{pmatrix},$$

$$D^- := \begin{pmatrix} -\infty & -2 & -3 & 3 & 2 & -1 & 1 & +\infty \\ +\infty & -1 & 2 & 3 & -2 & 1 & -3 & -\infty \end{pmatrix},$$

$$E^+ := \begin{pmatrix} -\infty & -1 & 1 & -2 & -3 & 2 & 3 & +\infty \\ +\infty & 3 & -3 & -2 & 1 & -1 & 2 & -\infty \end{pmatrix},$$

$$E^- := \begin{pmatrix} -\infty & -3 & 3 & 2 & -1 & 1 & -2 & +\infty \\ +\infty & 1 & -1 & 2 & 3 & -2 & -3 & -\infty \end{pmatrix},$$

$$F^+ := \begin{pmatrix} -\infty & -2 & 1 & 3 & -3 & -1 & 2 & +\infty \\ +\infty & -1 & -2 & 2 & -3 & 3 & 1 & -\infty \end{pmatrix},$$

$$F^- := \begin{pmatrix} -\infty & 1 & 2 & -2 & 3 & -3 & -1 & +\infty \\ +\infty & 2 & -1 & -3 & 3 & 1 & -2 & -\infty \end{pmatrix},$$

$$G^+ := \begin{pmatrix} -\infty & -2 & 1 & 3 & -3 & 2 & -1 & +\infty \\ +\infty & -1 & -3 & -2 & 2 & 3 & 1 & -\infty \end{pmatrix},$$

$$G^- := \begin{pmatrix} -\infty & 1 & 3 & 2 & -2 & -3 & -1 & +\infty \\ +\infty & 2 & -1 & -3 & 3 & -2 & 1 & -\infty \end{pmatrix},$$

$$H^+ := \begin{pmatrix} -\infty & 1 & 3 & -3 & 2 & -2 & -1 & +\infty \\ +\infty & 2 & -1 & -3 & -2 & 3 & 1 & -\infty \end{pmatrix},$$

$$H^- := \begin{pmatrix} -\infty & -2 & 1 & 3 & 2 & -3 & -1 & +\infty \\ +\infty & -1 & -3 & 3 & -2 & 2 & 1 & -\infty \end{pmatrix},$$

$$I^+ := \begin{pmatrix} -\infty & 1 & 3 & -3 & -1 & 2 & -2 & +\infty \\ +\infty & 2 & -1 & -2 & -3 & 3 & 1 & -\infty \end{pmatrix},$$

$$I^- := \begin{pmatrix} -\infty & -2 & 1 & 2 & 3 & -3 & -1 & +\infty \\ +\infty & -1 & -3 & 3 & 1 & -2 & 2 & -\infty \end{pmatrix},$$

$$J^+ := \begin{pmatrix} -\infty & -2 & -1 & 1 & 3 & -3 & 2 & +\infty \\ +\infty & -1 & 2 & -3 & -2 & 3 & 1 & -\infty \end{pmatrix},$$

$$J^- := \begin{pmatrix} -\infty & 1 & -2 & 3 & 2 & -3 & -1 & +\infty \\ +\infty & 2 & 1 & -1 & -3 & 3 & -2 & -\infty \end{pmatrix},$$

$$K^+ := \begin{pmatrix} -\infty & 1 & -2 & 3 & -3 & 2 & -1 & +\infty \\ +\infty & 2 & 3 & 1 & -1 & -3 & -2 & -\infty \end{pmatrix},$$

$$K^- := \begin{pmatrix} -\infty & -2 & -3 & -1 & 1 & 3 & 2 & +\infty \\ +\infty & -1 & 2 & -3 & 3 & -2 & 1 & -\infty \end{pmatrix},$$

$$L^+ := \begin{pmatrix} -\infty & 1 & -2 & 3 & -3 & -1 & 2 & +\infty \\ +\infty & 2 & -3 & 3 & 1 & -1 & -2 & -\infty \end{pmatrix},$$

$$L^- := \begin{pmatrix} -\infty & -2 & 3 & -3 & -1 & 1 & 2 & +\infty \\ +\infty & -1 & 2 & -3 & 3 & 1 & -2 & -\infty \end{pmatrix},$$

$$M^+ := \begin{pmatrix} -\infty & -2 & 3 & -3 & 2 & -1 & 1 & +\infty \\ +\infty & -1 & 2 & 3 & 1 & -3 & -2 & -\infty \end{pmatrix},$$

$$M^- := \begin{pmatrix} -\infty & 1 & -2 & -3 & -1 & 3 & 2 & +\infty \\ +\infty & 2 & -3 & 3 & -2 & 1 & -1 & -\infty \end{pmatrix},$$

$$N^+ := \begin{pmatrix} -\infty & 1 & -2 & -1 & 3 & -3 & 2 & +\infty \\ +\infty & 2 & -3 & -2 & 3 & 1 & -1 & -\infty \end{pmatrix},$$

$$N^- := \begin{pmatrix} -\infty & -2 & 3 & 2 & -3 & -1 & 1 & +\infty \\ +\infty & -1 & 2 & 1 & -3 & 3 & -2 & -\infty \end{pmatrix},$$

$$O^+ := \begin{pmatrix} -\infty & -1 & 1 & 3 & -2 & -3 & 2 & +\infty \\ +\infty & 3 & -1 & 2 & -3 & -2 & 1 & -\infty \end{pmatrix},$$

$$O^- := \begin{pmatrix} -\infty & -3 & 1 & -2 & 3 & 2 & -1 & +\infty \\ +\infty & 1 & -1 & -3 & 2 & 3 & -2 & -\infty \end{pmatrix},$$

$$P^+ := \begin{pmatrix} -\infty & -1 & 3 & 1 & -2 & -3 & 2 & +\infty \\ +\infty & 3 & 2 & -3 & -2 & 1 & -1 & -\infty \end{pmatrix},$$

$$P^- := \begin{pmatrix} -\infty & -3 & -2 & 3 & 2 & -1 & 1 & +\infty \\ +\infty & 1 & -3 & -1 & 2 & 3 & -2 & -\infty \end{pmatrix},$$

$$Q^+ := \begin{pmatrix} -\infty & 1 & 3 & -2 & -3 & 2 & -1 & +\infty \\ +\infty & 2 & 3 & -1 & -3 & -2 & 1 & -\infty \end{pmatrix},$$

$$Q^- := \begin{pmatrix} -\infty & -2 & -3 & 1 & 3 & 2 & -1 & +\infty \\ +\infty & -1 & -3 & 2 & 3 & -2 & 1 & -\infty \end{pmatrix},$$

$$R^+ := \begin{pmatrix} -\infty & 1 & 3 & -2 & -3 & -1 & 2 & +\infty \\ +\infty & 2 & -3 & 3 & -1 & -2 & 1 & -\infty \end{pmatrix},$$

$$R^- := \begin{pmatrix} -\infty & -2 & 3 & -3 & 1 & 2 & -1 & +\infty \\ +\infty & -1 & -3 & 2 & 3 & 1 & -2 & -\infty \end{pmatrix},$$

$$S^+ := \begin{pmatrix} -\infty & 1 & 3 & -2 & -1 & -3 & 2 & +\infty \\ +\infty & 2 & -3 & -2 & 3 & -1 & 1 & -\infty \end{pmatrix},$$

$$S^- := \begin{pmatrix} -\infty & -2 & 3 & 2 & -3 & 1 & -1 & +\infty \\ +\infty & -1 & -3 & 2 & 1 & 3 & -2 & -\infty \end{pmatrix},$$

$$T^+ := \begin{pmatrix} -\infty & 1 & 3 & -3 & -2 & -1 & 2 & +\infty \\ +\infty & 2 & -3 & -1 & -2 & 3 & 1 & -\infty \end{pmatrix},$$

$$T^- := \begin{pmatrix} -\infty & -2 & 3 & 1 & 2 & -3 & -1 & +\infty \\ +\infty & -1 & -3 & 3 & 2 & 1 & -2 & -\infty \end{pmatrix},$$

$$U^+ := \begin{pmatrix} -\infty & 1 & -1 & 3 & -2 & -3 & 2 & +\infty \\ +\infty & 2 & -3 & -2 & 1 & 3 & -1 & -\infty \end{pmatrix},$$

$$U^- := \begin{pmatrix} -\infty & -2 & 3 & 2 & -1 & -3 & 1 & +\infty \\ +\infty & -1 & 1 & -3 & 2 & 3 & -2 & -\infty \end{pmatrix}.$$

23.3. **The diagram $\Gamma(\mathcal{D})$.** Four vertices ($X, Y, A^\pm$) have valence 8

- X is connected to $F^\pm, G^\pm, H^\pm, I^\pm$;
- Y is connected to $B^\pm, C^\pm, D^\pm, E^\pm$:
- A^+ is connected to $A^-, B^+, L^\pm, J^+, K^+, M^+, N^+$;

Four vertices ($B^\pm, O^\pm$) have valence 7

- B^+ is connected to $Y, A^+, C^+, D^+, E^+, K^+, M^+$;
- O^+ is connected to $C^+, J^+, Q^+, R^+, S^+, U^+, K^-$.

Six vertices ($C^\pm, J^\pm, K^\pm$) have valence 6

- C^+ is connected to $Y, B^+, D^+, E^+, O^+, Q^+$;
- J^+ is connected to $A^+, H^+, N^+, O^+, S^+, T^+$;
- K^+ is connected to $A^+, B^+, G^+, M^+, O^-, R^-$.

Four vertices ($D^\pm, F^\pm$) have valence 5

- D^+ is connected to Y, B^+, C^+, E^+, P^+;
- F^+ is connected to X, I^+, L^+, R^+, T^+.

Twelve vertices ($E^\pm, G^\pm, L^\pm, M^\pm, P^\pm, Q^\pm$) have valence 4

- E^+ is connected to Y, B^+, C^+, D^+;
- G^+ is connected to X, H^+, K^+, Q^+;
- L^+ is connected to $A^\pm, F^+, L^-$;
- M^+ is connected to A^+, B^+, K^+, P^-;
- P^+ is connected to D^+, N^+, U^+, M^-;
- Q^+ is connected to C^+, G^+, O^+, Q^-.

Six vertices ($H^\pm, N^\pm, R^\pm$) have valence 3

- H^+ is connected to X, G^+, J^+;
- N^+ is connected to A^+, J^+, P^+;
- R^+ is connected to F^+, O^+, K^-.

Eight vertices ($I^\pm, S^\pm, T^\pm, U^\pm$) have valence 2

- I^+ is connected to X, F^+;
- S^+ is connected to J^+, O^+;
- T^+ is connected to F^+, J^+;
- U^+ is connected to O^+, P^+.

The default of the diagram is 99.

23.4. **Up to height** 4**.** There are 44 pure cycles of each type and height 1. Each has length 7. Therefore there are 264 vertices with $H_t = 2, H_b = 4$ and 264 vertices with $H_t = 4, H_b = 2$. For each $\ell = 1, 2, 3, 4, 5, 6$, there are 44 pure cycles of each type, height 3, length ℓ. Therefore there are 660 vertices with $H_t = H = 4$, and 660 vertices with $H_b = H = 4$.

In view of the default of $\Gamma(\mathcal{D})$, there are 198 vertices with $H_b = H_t = 4$, 462 with $H_b = 4, H_t = 6$ and 462 with $H_t = 4, H_b = 6$.

23.5. **Cycles of height** 5**.** Among the 462 vertices V with $H_b = 4, H_t = 6$,

- the length of the top cycle (of height 5) through V is equal to 1 in 160 cases;
- the length of the top cycle (of height 5) through V is equal to 2 in 116 cases;
- the length of the top cycle (of height 5) through V is equal to 3 in 92 cases;
- the length of the top cycle (of height 5) through V is equal to 4 in 60 cases;
- the length of the top cycle (of height 5) through V is equal to 5 in 34 cases.

There are 96 pure cycles of top type, height 5, length 2. 20 have two vertices of height 4. The other 76 have a vertex of height 4 and a vertex of height 6. Denoting by C the cycle of bottom type through this vertex

- the length of C is equal to 1 in 46 cases;
- the length of C is equal to 2 in 22 cases;
- the length of C is equal to 3 in 7 cases;
- the length of C is equal to 4 in 1 case.

There are 59 pure cycles of top type, height 5, length 3.

- 9 have three vertices of height 4;
- 15 have two vertices of height 4 and one vertex of height 6;
- 35 have one vertex of height 4 and two vertices of height 6.

There are 32 pure cycles of top type, height 5, length 4.

- 2 have four vertices of height 4;
- 6 have three vertices of height 4 and one vertex of height 6;
- 10 have two vertices of height 4 and two vertices of height 6;
- 14 have one vertex of height 4 and three vertices of height 6.

There are 16 pure cycles of top type, height 5, length 5.

- 1 have five vertices of height 4;
- 1 have four vertices of height 4 and one vertex of height 6;
- 3 have three vertices of height 4 and two vertices of height 6;
- 5 have two vertex of height 4 and three vertices of height 6:
- 6 have one vertex of height 4 and four vertices of height 6.

Summing up, there are 275 vertices with $H_t = H = 6$. Denoting by ℓ the length of the bottom cycle through these vertices

- $\ell = 1$ in 131 cases;
- $\ell = 2$ in 75 cases;
- $\ell = 3$ in 49 cases;
- $\ell = 4$ in 20 cases;

23.6. **Cycles of height** 7**.** When $\ell = 1$, the bottom cycle has height 7. When $\ell > 1$, the height may be equal to 7 or 5.

There are 40 vertices with $H_t = H_b = 6$, 235 with $H_t = 6, H_b = 8$, and 235 with $H_b = 6, H_t = 8$. Among the vertices V with $H_t = 6, H_b = 8$, the length of the bottom cycle of height 7 through V is equal to

- 1 in 131 cases;
- 2 in 59 cases;
- 3 in 33 cases;
- 4 in 12 cases;

There are 45 cycles of bottom type, height 7, length 2. Among these cycles

- 14 have two vertices of height 6;
- 26 have one vertex of height 6 and one vertex of height 8, with the top cycle through this last vertex of length 1 (hence height 9);
- 4 have one vertex of height 6 and one vertex of height 8, with the top cycle through this last vertex of length > 1, height 9;
- 1 have one vertex of height 6 and one vertex of height 8, with the top cycle through this last vertex of height 7; this vertex has thus $H_t = H_b = 8$.

There are 16 cycles of bottom type, height 7, length 3. Among these cycles

- 5 have three vertices of height 6;
- 7 have two vertices of height 6 and one vertex of height 8 with the top cycle through this last vertex of length 1 (hence height 9);
- 4 have one vertex of height 6 and two vertices of height 8. In all cases, the top cycle through one of the two vertices of height 8 has length 1. In two cases, the top cycle through the other vertex of height 8 has length 2, with the new vertex of height 10 but inessential. In the other two cases, the top cycle through the other vertex of height 8 has length 3, height 9.

There are 6 cycles of bottom type, height 7, length 4. Among these cycles

- 2 have three vertices of height 6 and one vertex of height 8;
- 2 have two vertices of height 6 and two vertices of height 8;
- 2 have one vertex of height 6 and three vertices of height 8.

There appears to be 8 cycles of height 9, top type and length > 1

- Three of them have length 2 and contain one vertex of height 8 and one inessential vertex of height 10. The vertex of height 8 is linked to P^+ in one case, to E^-/P^- in the second case, to T^+ in the third case. The corresponding inessential vertices are

$$\begin{pmatrix} -\infty & -1 & +\infty & 3 & 1 & 2 & -2 & -3 \\ +\infty & -\infty & 3 & 2 & -1 & -3 & 1 & -2 \end{pmatrix},$$

$$\begin{pmatrix} -\infty & -3 & -2 & +\infty & 3 & 1 & 2 & -1 \\ +\infty & -\infty & 1 & -2 & -3 & -1 & 3 & 2 \end{pmatrix},$$

$$\begin{pmatrix} -\infty & 1 & +\infty & 3 & 2 & -3 & -2 & -1 \\ +\infty & -\infty & 2 & 1 & -3 & -1 & 3 & -2 \end{pmatrix}.$$

- Three other cycles have length 2 and contain two vertices of height 8. These cycles are the middle cycles of monotonous chains linking G^- to H^+/S^+ (for one cycle), T^- to R^-/U^- (for the second cycle) and R^- to S^-. The vertices of height 8 on the G^- (resp. T^-, resp. R^-) side are

$$\begin{pmatrix} -\infty & 1 & +\infty & 3 & 2 & -2 & -1 & -3 \\ +\infty & -\infty & 2 & -1 & 1 & -3 & 3 & -2 \end{pmatrix},$$
$$\begin{pmatrix} -\infty & -2 & +\infty & 3 & 1 & 2 & -1 & -3 \\ +\infty & -\infty & -1 & 1 & -2 & -3 & 3 & 2 \end{pmatrix},$$
$$\begin{pmatrix} -\infty & -2 & +\infty & 3 & -3 & 1 & -1 & 2 \\ +\infty & -\infty & -1 & -2 & -3 & 2 & 3 & 1 \end{pmatrix}.$$

- One cycle has length 3 and contains two vertices of height 8 (linked to S^-, T^-) and one inessential vertex of height 10 equal to

$$\begin{pmatrix} -\infty & -2 & +\infty & 3 & 1 & -1 & 2 & -3 \\ +\infty & -\infty & -1 & -2 & -3 & 1 & 3 & 2 \end{pmatrix}.$$

- The last cycle has length 3 and contains three vertices of height 8 linked respectively to R^+, S^+, T^+. The vertex linked to R^+ is

$$\begin{pmatrix} -\infty & 1 & +\infty & 3 & -2 & -1 & 2 & -3 \\ +\infty & -\infty & 2 & 1 & -3 & 3 & -1 & -2 \end{pmatrix}.$$