SOME VARIANTS OF ORPONEN'S THEOREM ON VISIBLE PARTS OF FRACTAL SETS

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Abstract. It was recently established by T. Orponen that the visible parts from almost every direction of a compact subset of $\mathbb{R}^n$ have Hausdorff dimension at most $n - \frac{1}{50n}$.

In this note, we refine Orponen's argument in order to show that the visible parts from almost every direction of a compact subset of $\mathbb{R}^n$ have Hausdorff dimension at most $n - \min\{\frac{1}{5}, \frac{1}{n+2}\}$.

Moreover, we also show that some classes of dynamically defined Cantor sets $K \subset \mathbb{R}^n$ with Hausdorff dimension $d > \max\{\sqrt{3}, \frac{(n-1)+\sqrt{(n-1)(n+3)}}{2}\}$ have visible parts of Hausdorff dimension at most $\max\{\frac{3d+3}{d+3}, \frac{(n+1)d+(n-1)}{d+2}\}$ from almost every direction.

1. Introduction

Let K be a compact subset of the Euclidean space $\mathbb{R}^n$, $n \geq 2$. Intuitively, the *visible part* $\text{Vis}_e(K)$ of K in the direction $e \in S^{n-1}$ is the subset of K consisting of the points which are first hit by a light beam travelling in the direction e emanating from a certain affine hyperplane orthogonal to e.

More concretely, if $\pi_e : \mathbb{R}^n \to e^\perp$ denotes the orthogonal projection to the hyperplane $e^\perp$ orthogonal to e and $\langle ., . \rangle$ stands for the usual Euclidean inner product, then $\text{Vis}_e(K)$ is the collection of $\leq_e$-minimal points of K where $\leq_e$ is the partial order defined by $x \leq_e y$ if and only if $\pi_e(x) = \pi_e(y)$ and $\langle x, e \rangle \leq \langle y, e \rangle$.

In general, the visible parts $\text{Vis}_e(K)$ are Borel sets because they are the graphs of lower semi-continuous functions, cf. [4, Remark 2.2 (a)].

By definition, $\pi_e(\text{Vis}_e(K)) = \pi_e(K)$ for all $e \in S^{n-1}$. Therefore, Mattila's extension of Marstrand's theorem [7] provides the following *lower bound* on the Hausdorff dimension of typical visible parts:

$$\dim_H(\text{Vis}_e(K)) \geq \min\{\dim_H(K), n-1\}$$

for Lebesgue almost every $e \in S^{n-1}$.

The *visibility conjecture* asserts that the converse inequality is true, i.e., if $\dim_H(K) > n-1$, then $\dim_H(\text{Vis}_e(K)) = n-1$ for Lebesgue almost every $e \in S^{n-1}$ (see, e.g., [8, Problem 11]).

It is known that this conjecture admits a positive answer for several particular classes of compact subsets of $\mathbb{R}^n$ (cf. [4], [2] and [1]). Furthermore, we know that if $K \subset \mathbb{R}^n$ is a compact subset with d-Hausdorff measure $0 < \mathcal{H}^d(K) < \infty$, then the d-Hausdorff measure of $\text{Vis}_e(K)$ is zero for Lebesgue almost every $e \in S^{n-1}$ (see [5, Theorem 1.1]).

More recently, T. Orponen [9] obtained an *unconditional* estimate on the Hausdorff dimension of typical visible parts of compact subsets K of $\mathbb{R}^n$: in a nutshell, he proved that $\dim_H(\text{Vis}_e(K)) \leq n - \frac{1}{50n}$ for Lebesgue almost every $e \in S^{n-1}$.

In this note, we refine Orponen's methods to establish the following two results:

Date: March 5, 2020.

Theorem 1.1. *Let $K \subset \mathbb{R}^n$ be a compact subset. Then, for Lebesgue almost every $e \in S^{n-1}$, the Hausdorff dimension of $Vis_e(K)$ is at most $n - \min\{\frac{1}{5}, \frac{1}{n+2}\}$.*

Theorem 1.2. *Let $K \subset \mathbb{R}^n$ be a product of C^2-dynamically defined Cantor sets of the real line or a self-similar set defined by a finite collection of Euclidean similarities verifying the open set condition. If the Hausdorff dimension of K is $\dim_H(K) > \max\{\sqrt{3}, \frac{(n-1)+\sqrt{(n-1)(n+3)}}{2}\}$, then, for Lebesgue almost every $e \in S^{n-1}$, the Hausdorff dimension of $Vis_e(K)$ is at most $\max\{\frac{3d+3}{d+3}, \frac{(n+1)d+(n-1)}{d+2}\}$.*

The remainder of this note is divided into two sections: its first half contains the proof of Theorem 1.1 and its second half is devoted to the proof of Theorem 1.2.

2. Visible parts of general compact subsets

Let K be a compact subset of $\mathbb{R}^n$, $n \geq 2$. Up to rescaling, we can (and do) assume that $K \subset [0,1]^n$. Since the conclusion of Theorem 1.1 always holds when K has Hausdorff dimension $\leq n - \min\{\frac{1}{5}, \frac{1}{n+2}\}$, we can (and do) also assume that

$$n - \min\left\{\frac{1}{5}, \frac{1}{n+2}\right\} < d := \dim_H(K) \leq n. \tag{2.1}$$

2.1. **Some preliminaries.** Recall that the s-dimensional Hausdorff measure at scale $0 < \rho \leq \infty$ of a subset $E \subset \mathbb{R}^n$ is

$$\mathcal{H}^s_\rho(E) := \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}(U_i)^s : E \subset \bigcup_{i\geq 1} U_i \text{ and } \operatorname{diam}(U_i) < \delta \ \forall\, i \geq 1\right\}$$

and the s-dimensional Hausdorff measure of E is $\mathcal{H}^s(E) = \lim\limits_{\rho\to 0} \mathcal{H}^s_\rho(E)$, so that the Hausdorff dimension of E is

$$\dim_H(E) := \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

Recall also that a dyadic cube $Q \subset [0,1]^n$ is a cube of the form $Q = \prod\limits_{j=1}^{n} [\frac{i_j}{2^N}, \frac{i_j+1}{2^N}]$ for some $N \in \mathbb{N}$ and $(i_1, \dots, i_n) \in \{1, \dots, 2^N - 1\}^n$. In the sequel, the collection of dyadic cubes with sides of fixed size 2^{-N} is denoted by $\mathcal{D}_{2^{-N}}$.

In [9, Lemma A.1], Orponen showed the following version of Frostman's lemma:

Lemma 2.1 (Orponen)**.** *Let $E \subset [0,1]^n$ be a compact subset. Then, there exists a Radon measure μ supported on E and a constant $0 < C = C(n) < \infty$ such that $\mu(B(x,r)) \leq Cr^s$ for all $x \in \mathbb{R}^n$, $r > 0$, and*

$$\mu(Q) \geq C^{-1} \min\{\mathcal{H}^s_\infty(E \cap Q), \mathcal{H}^n(Q)\}$$

for all dyadic cube $Q \subset [0,1]^n$.

Similarly to Orponen [9], our long-term goal is to apply this lemma to estimate the Hausdorff dimension of visible parts in typical directions.

For this sake, we fix first some rational parameters

$$n - \min\left\{\frac{1}{5}, \frac{1}{n+2}\right\} < n - \varepsilon_0 < s_0'' < s_0' < s_0 < d \leq n, \tag{2.2}$$

$$\alpha := \min\left\{s_0'' - 1, 2 - \frac{s_0' - (n-1)}{2}\right\}, \tag{2.3}$$

$$\frac{\varepsilon_0 n}{2} < \frac{\varepsilon_1}{2} < \min\{s_0'' - 1, 1\} - \frac{\varepsilon_0 n}{2} - 2\varepsilon_0, \tag{2.4}$$

and

$$0 < \varepsilon_* < \min\left\{s_0' + \varepsilon_0 - n, \frac{2}{3}\left(\min\{s_0'' - 1, 1\} - \frac{\varepsilon_0 n}{2} - 2\varepsilon_0 - \frac{\varepsilon_1}{2}\right)\right\}. \tag{2.5}$$

Note that these conditions are mutually compatible: indeed, our assumption (2.1) allows us to choose s_0, s_0', s_0'' and ε_0 in (2.2); since $\varepsilon_0 < \min\left\{\frac{1}{5}, \frac{1}{n+2}\right\}$ and $s_0' > n - \varepsilon_0$, we can select ε_1 in (2.4) and ε_* in (2.5).

Now, we use Lemma 2.1 to get μ supported on K such that

$$\mu(B(x,r)) \leq Cr^{s_0} \tag{2.6}$$

for all $x \in \mathbb{R}^n$ and $r > 0$, and

$$\mu(Q) \geq C^{-1} \min\{\mathcal{H}^{s_0}_\infty(K \cap Q), \mathcal{H}^n(Q)\} \tag{2.7}$$

for all dyadic cube $Q \subset [0,1]^n$ (where $0 < C = C(n) < \infty$ is a constant).

Recall that (2.6) implies that the s_0'-*energy* of μ is finite, i.e.,

$$I_{s_0'}(\mu) := \int\int \frac{d\mu(x)\, d\mu(y)}{|x-y|^{s_0'}} < \infty. \tag{2.8}$$

Remark 2.2. *For later reference, let us remind that the s-energy of a measure θ can be expressed in terms of the Fourier transform as*

$$I_s(\theta) = \int\int \frac{d\theta(x)\, d\theta(y)}{|x-y|^s} = c_1(s,n) \int |\widehat{\theta}(\xi)|^2 \cdot |\xi|^{s-n}\, d\mathcal{H}^n(\xi)$$

where $0 < c_1(s,n) < \infty$ is a constant.

In the sequel, $\delta = 2^{-N}$, $N \in \mathbb{N}$, is an arbitrary (small) dyadic scale such that δ^{ε_0} is also a dyadic scale.

2.2. **Contribution of light cubes.** We say that a dyadic cube $Q \in \mathcal{D}_\delta$ is δ-*light* when $\mu(Q) \leq \delta^{n+\varepsilon_*}$. The portion of K contained in δ-light cubes is denoted by $K_{\delta,\text{light}}$. Since $\mathcal{D}_\delta$ has cardinality δ^{-n}, it follows from (2.7) that:

Lemma 2.3. $\mathcal{H}^{s_0}_\infty(K_{\delta,light}) \leq C(n) \cdot \delta^{\varepsilon_*}$.

In particular, this lemma says that we can safely focus on the δ-*heavy* portion $K_{\delta,\text{heavy}} := K \setminus K_{\delta,\text{light}}$ of K.

2.3. **Exceptional directions.** Given a dyadic cube $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$, the restriction of μ to Q is denoted by μ_Q. The set of δ-*exceptional directions* associated to Q is

$$E_{\delta,Q} := \left\{e \in S^{n-1} : \int_{e^\perp} |\widehat{\mu_Q}(\zeta)|^2 \cdot |\zeta|^{s_0' - (n-1)}\, d\mathcal{H}^{n-1}(\zeta) \geq \delta^{-\varepsilon_1}\right\}.$$

Since $I_{s_0'}(\mu_Q) \leq I_{s_0'}(\mu)$, it follows from (2.8) and a change of variables to polar coordinates in Remark 2.2 that:

Lemma 2.4. $\mathcal{H}^{n-1}(E_{\delta,Q}) \leq c_2(s_0', n) I_{s_0'}(\mu)\delta^{\varepsilon_1}$ *for all* $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$.

2.4. **Good and bad lines.** Denote by $\mathcal{L}_e$ the space of lines parallel to $e \in S^{n-1}$. Given a dyadic cube $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ intersecting K, the set $\mathcal{L}_{e,\delta,\text{bad},Q}$ of δ-*bad lines* in direction e associated to Q consists of all lines $\ell \in \mathcal{L}_e$ disjoint from $K \cap Q$ whose 2δ-neighborhood $\ell(2\delta)$ satisfy

$$\#\{R \in \mathcal{D}_\delta : R \subset Q, R \cap K \neq \emptyset, R \text{ is not light}, R \cap \ell(2\delta) \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

We say that $\ell \in \mathcal{L}_e$ is a δ-*good line* in the direction e whenever $\ell \notin \mathcal{L}_{e,\delta,\text{bad},Q}$ for all $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ intersecting K. The collection of δ-good lines in the direction e is denoted by $\mathcal{L}_{e,\delta,\text{good}}$ and we define

$$L_{e,\delta,\text{good}} := \bigcup_{\ell \in \mathcal{L}_{e,\delta,\text{good}}} \ell.$$

Lemma 2.5. $\mathcal{H}_\delta^{s_0'}(\mathit{Vis}_e(K) \cap K_{\delta,heavy} \cap L_{e,\delta,good}) \leq \delta^{\varepsilon_*}$ *for all δ sufficiently small.*

Proof. Let us use a collection $\mathcal{T}_{e,\delta}$ tubes of width δ whose bases are perpendicular to e in order to cover $[0,1]^n$. Since $\#\mathcal{T}_{e,\delta} \leq c_3(n)\delta^{-(n-1)}$, our task is reduced to prove that, for each $T \in \mathcal{T}_{e,\delta}$, the minimal number $N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta)$ of δ-balls needed to cover $\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T$ is at most

$$N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq c_5(n)\delta^{\varepsilon_0 - 1}.$$

Indeed, the estimates above imply that

$$\mathcal{H}_\delta^{s_0'}(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}}) \leq c_3(n)c_5(n)\delta^{-(n-1)}\delta^{\varepsilon_0-1}\delta^{s_0'} \leq \delta^{\varepsilon_*}$$

for all δ sufficiently small thanks to the fact that $s_0' + \varepsilon_0 - n > \varepsilon_*$ (cf. (2.5)).

In order to estimate $N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta)$ for a given $T \in \mathcal{T}_{e,\delta}$, we consider two scenarios:

(i) for all $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ intersecting K, one has

$$\#\{R \in \mathcal{D}_\delta : R \subset Q, R \cap K \neq \emptyset, R \text{ is not light}, R \cap T \neq \emptyset\} < \delta^{2\varepsilon_0 - 1};$$

(ii) there exists $Q_1 \in \mathcal{D}_{\delta^{\varepsilon_0}}$ intersecting K with

$$\#\{R \in \mathcal{D}_\delta : R \subset Q_1, R \cap K \neq \emptyset, R \text{ is not light}, R \cap T \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

In the first scenario, we have that $N(\text{Vis}_e(K) \cap K_{\delta,\text{heavy}} \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq \delta^{\varepsilon_0 - 1}$ simply because T can meet at most $\delta^{-\varepsilon_0}$ dyadic cubes $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$.

In the second scenario, we take Q_1 to be a $\leq_e$-minimal dyadic cube with the property described in (ii) (in the sense that Q_1 minimizes $\inf\{\langle x, e\rangle : x \in Q_1\}$ among all dyadic cubes in (ii)). Since the 2δ-neighborhood of any line $\ell \subset T$ contains T, we also have

$$\#\{R \in \mathcal{D}_\delta : R \subset Q_1, R \cap K \neq \emptyset, R \text{ is not light}, R \cap \ell(2\delta) \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

Therefore, it follows from the definition of δ-good line that any $\ell \in \mathcal{L}_{e,\delta,\text{good}}$ included in T must intersect $K \cap Q_1$.

We affirm that

$$\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T \cap Q = \emptyset$$

for any dyadic cube $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ with $\inf\{\langle x, e\rangle : x \in Q\} > \sup\{\langle y, e\rangle : y \in Q_1\}$. In fact, if $x \in \text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T \cap Q$, then $\pi_e(x) = \pi_e(y)$ for some $y \in Q_1$. Since $\langle x, e\rangle > \langle y, e\rangle$, one would get $x \notin \text{Vis}_e(K)$, a contradiction.

Hence, $\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T$ is covered by the collection of dyadic cubes $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ with $\inf\{\langle x, e\rangle : x \in Q\} \leq \sup\{\langle y, e\rangle : y \in Q_1\}$. Now, we observe that

- the number of dyadic cubes $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ intersecting T with

  $$\inf\{\langle z, e\rangle : z \in Q_1\} \leq \inf\{\langle x, e\rangle : x \in Q\} \leq \sup\{\langle y, e\rangle : y \in Q_1\}$$

  is bounded by an absolute constant $c_4(n)$; for each of them, we will use the crude bound $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq \delta^{\varepsilon_0 - 1}$ coming from the fact that $Q \cap T$ can be covered using at most $\delta^{\varepsilon_0 - 1}$ balls of radius δ;

- any dyadic cube $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ intersecting $T \cap K$ with
$$\inf\{\langle z, e\rangle : z \in Q_1\} > \inf\{\langle x, e\rangle : x \in Q\}$$
satisfies
$$\#\{R \in \mathcal{D}_\delta : R \subset Q_1, R \cap K \neq \emptyset, R \text{ is not light }, R \cap \ell(2\delta) \neq \emptyset\} \leq \delta^{2\varepsilon_0 - 1}$$
because of the $\leq_e$-minimality of Q_1; the number of such cubes Q is at most $\leq \delta^{-\varepsilon_0}$ because T meets at most $\delta^{-\varepsilon_0}$ dyadic cubes $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$.

By combining the estimates above, we conclude that
$$N(\mathrm{Vis}_e(K) \cap K_{\delta,\mathrm{heavy}} \cap L_{e,\delta,\mathrm{good}} \cap T, \delta) \leq c_4(n)\delta^{\varepsilon_0 - 1} + \delta^{-\varepsilon_0}\delta^{2\varepsilon_0 - 1} = c_5(n)\delta^{\varepsilon_0-1}.$$
This completes the proof. □

2.5. **Typical visible parts in bad lines.** The last step towards the proof of Theorem 1.1 is the following estimate:

Lemma 2.6. *Let $Q \in \mathcal{D}_{\delta^{\varepsilon_0}}$ be a dyadic cube intersecting K, consider a direction $e \notin E_{\delta,Q}$, and denote $L_{e,\delta,bad,Q} := \bigcup_{\ell \in \mathcal{L}_{e,\delta,bad,Q}} \ell$. Then,*
$$\mathcal{H}^{s_0'-1}_\infty(\pi_e(L_{e,\delta,bad,Q})) \leq \delta^{\varepsilon_* + \varepsilon_0 n}$$
for all δ sufficiently small.

Proof. By contradiction, suppose that $\mathcal{H}^{s_0'-1}_\infty(\pi_e(L_{e,\delta,\mathrm{bad},Q})) \geq \delta^{\varepsilon_* + \varepsilon_0 n}$. By Orponen's version of Frostman's lemma (cf. Lemma 2.1), we have a *probability* measure ν supported on $H_{e,\delta,Q} := \pi_e(L_{e,\delta,\mathrm{bad},Q})$ such that
$$\nu(B(x,r)) \leq C(n-1)\delta^{-\varepsilon_* - \varepsilon_0 n} r^{s_0'-1}$$
for all $x \in H$ and $r > 0$. Thus, our choice of $\alpha \leq s_0'' - 1 < s_0' - 1$ in (2.3) (and Remark 2.2) means that the α-energy of ν satisfies
$$c_1(\alpha, n-1)\int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^\alpha \, d\xi = I_\alpha(\nu) \leq c_6(s_0'', s_0', n)\delta^{-\varepsilon_* - \varepsilon_0 n}. \tag{2.9}$$

Next, we observe that, by definition, any line $\ell \in \mathcal{L}_{e,\delta,\mathrm{bad},Q}$ misses $K \cap Q$. Therefore, $\mu_{Q,e} := (\pi_e)_*(\mu_Q)$ and ν have disjoint supports. Hence, if we fix a non-negative smooth bump function φ on $e^\perp \simeq \mathbb{R}^{n-1}$ with total integral one and $\varphi(0) = 1$, then
$$\begin{aligned} 0 &= \int \varphi_\eta * \mu_{Q,e} \, d\nu = \int \widehat{\varphi}(\eta\xi)\widehat{\mu_{Q,e}}(\xi)\overline{\widehat{\nu}(\xi)} \, d\xi \\ &= \int (1 - \widehat{\varphi}(c_7(n)\delta\xi))\widehat{\varphi}(\eta\xi)\widehat{\mu_{Q,e}}(\xi)\overline{\widehat{\nu}(\xi)} \, d\xi + \int \widehat{\varphi}(c_7(n)\delta\xi)\widehat{\varphi}(\eta\xi)\widehat{\mu_{Q,e}}(\xi)\overline{\widehat{\nu}(\xi)} \, d\xi \\ &:= A_2 - A_1 \end{aligned}$$
for all $0 < \eta \ll \delta$, where $\varphi_\eta(x) = \varphi(\eta x)/\eta^{n-1}$.

In the sequel, we will reach a contradiction with the identity in the previous paragraph by showing that $|A_2| < |A_1|$. For this sake, we observe that $\widehat{\varphi}$ is a bounded Lipschitz function with $\widehat{\varphi}(0) = 1$, so that $|1 - \widehat{\varphi}(c_7(n)\delta\xi)| \leq c_8(n)\delta|\xi|$ and, *a fortiori*,
$$|A_2| \leq c_8(n)\delta^{\frac{s_0'-(n-1)}{2} + \frac{\alpha}{2}} \left(\int |\widehat{\mu_{Q,e}}(\xi)|^2 \cdot |\xi|^{s_0'-(n-1)} \, d\xi\right)^{1/2} \left(\int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^\alpha \, d\xi\right)^{1/2}$$
thanks to our choice of $\frac{s_0'-(n-1)}{2} + \frac{\alpha}{2} \leq 1$ in (2.3) and the Cauchy–Schwarz inequality. By plugging into the previous inequality the facts that our choices in (2.2) and (2.3) imply $\frac{s_0'-(n-1)}{2} + \frac{\alpha}{2} \geq \min\{s_0''-1, 1\}$, our assumption $e \notin E_{\delta,Q}$ allows (by definition)

to control $|\widehat{\mu_Q}(\xi)|$ ($= |\widehat{\mu_{Q,e}}(\xi)|$ for $\xi \in e^\perp$), and the α-energy of ν is controlled by (2.9), we derive that

$$A_2 \leq c_9(s_0'', s_0', n)\delta^{\min\{s_0''-1,1\}}\delta^{-\varepsilon_1/2}\delta^{-(\varepsilon_*+\varepsilon_0 n)/2}.$$

On the other hand, if we write

$$A_1 = \int \varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r)\, d\nu(r),$$

and we recall that ν is supported in $H_{e,\delta,Q} := \pi_e(L_{e,\delta,\text{bad},Q})$, then we can use the fact that $r \in H_{e,\delta,Q}$ means $\ell := \pi_e^{-1}(r) \in \mathcal{L}_{e,\delta,\text{bad},Q}$, i.e., $\ell(2\delta)$ meets at least $\delta^{2\varepsilon_0-1}$ dyadic cubes $R \in \mathcal{D}_\delta$ included in Q which are not light, to deduce that $\mu_Q(\ell(2\delta)) \geq \delta^{2\varepsilon_0-1+n+\varepsilon_*}$ and, *a fortiori*,

$$\varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r) \geq c_{10}(n)\delta^{2\varepsilon_0+\varepsilon_*}$$

for all $r \in H$ and $0 < \eta \ll \delta$. Therefore,

$$A_1 \geq c_{10}(n)\delta^{2\varepsilon_0+\varepsilon_*}$$

because ν is a probability measure on H.

At this point, we get the desired contradiction $A_1 > |A_2|$ for δ is sufficiently small because our choice (2.5) implies that $2\varepsilon_0 + \varepsilon_* < \min\{s_0''-1, 1\} - \frac{\varepsilon_1}{2} - \frac{\varepsilon_*+\varepsilon_0 n}{2}$. □

2.6. **End of the proof of Theorem 1.1.** Let us take a decreasing sequence of dyadic scales $\delta_j \to 0$ such that $\delta_j^{\varepsilon_0}$ also a dyadic scale. We define the set E_{δ_j} of δ_j-exceptional directions as

$$E_{\delta_j} := \bigcup_{Q\in\mathcal{D}_{\delta_j^{\varepsilon_0}}} E_{\delta_j,Q}.$$

Since $\#\mathcal{D}_\eta = \eta^{-n}$, it follows from Lemma 2.4 that

$$\mathcal{H}^{n-1}(E_{\delta_j}) \leq c_2(s_0', n) I_{s_0'}(\mu)\delta_j^{\varepsilon_1-\varepsilon_0 n}.$$

Therefore, our choice of $\varepsilon_1 > \varepsilon_0 n$ in (2.4) implies

$$\sum_{j=1}^\infty \mathcal{H}^{n-1}(E_{\delta_j}) < \infty,$$

so that the set

$$E = E(s_0, s_0', s_0'', \varepsilon_0, \varepsilon_1, \varepsilon_*) := \bigcap_{n=1}^\infty \bigcup_{j\geq n} E_{\delta_j}$$

has zero $\mathcal{H}^{n-1}$-measure.

We affirm that $\dim_H(\text{Vis}_e(K)) \leq s_0$ whenever $e \in S^{n-1}\setminus E$. In fact, an element $e \notin E$ belongs to finitely many E_{δ_j}'s, say $e \notin E_{\delta_j}$ for all $j \geq j_e$.

By Lemma 2.3, we have $\mathcal{H}^{s_0}_\infty(\text{Vis}_e(K) \cap K_{\delta_j,\text{light}}) \leq \mathcal{H}^{s_0}_\infty(K_{\delta_j,\text{light}}) \leq C(n)\cdot\delta_j^{\varepsilon_*}$ for all j. Also, by Lemma 2.5, $\mathcal{H}^{s_0'}_{\delta_j}(\text{Vis}_e(K) \cap K_{\delta_j,\text{heavy}} \cap L_{e,\delta_j,\text{good}}) \leq \delta_j^{\varepsilon_*}$ for all j sufficiently large. Moreover, $\mathcal{H}^{s_0'}_\infty\left(\text{Vis}_e(K) \cap K_{\delta_j,\text{heavy}} \cap \bigcup_{\substack{Q\in\mathcal{D}_{\delta_j^{\varepsilon_0}},\\ Q\cap K\neq\emptyset}} L_{e,\delta_j,\text{bad},Q}\right) \leq \delta_j^{\varepsilon_*}$ for all $j \geq j_e$ sufficiently large by Lemma 2.6 (and the fact that $\#\mathcal{D}_{\delta_j^{\varepsilon_0}} = \delta_j^{-\varepsilon_0 n}$).

By putting these three estimates together, we derive that if $e \notin E$, then

$$\mathcal{H}^{s_0}_\infty(\text{Vis}_e(K)) \leq (C(n)+2)\delta_j^{\varepsilon_*}$$

for all $j \geq j_e$ sufficiently large, and, consequently, $\dim_H(\mathrm{Vis}_e(K)) \leq s_0$ for all $e \notin E(s_0, s_0', s_0'', \varepsilon_0, \varepsilon_1, \varepsilon_*)$.

Since $s_0, s_0', s_0'', \varepsilon_0, \varepsilon_1, \varepsilon_*$ are arbitrary rational parameters satisfying (2.2), (2.3), (2.4) and (2.5), we conclude that

$$\dim_H(\mathrm{Vis}_e(K)) \leq n - \min\left\{\frac{1}{5}, \frac{1}{n+2}\right\}$$

for Lebesgue almost every $e \in S^{n-1}$.

3. Typical visible parts of dynamical Cantor sets

In this section, we revisit Orponen's method described above in order to establish Theorem 1.2.

3.1. **Some preliminaries.** It is well-known (see, e.g., [6] and [3]) that the products of C^2-dynamically defined Cantor sets of the real line and the self-similar sets given by a finite collection of Euclidean similarities verifying the open set condition defined a class of compact subsets $K \subset \mathbb{R}^n$ with the following properties:

- K supports a measure μ equivalent to $\mathcal{H}^d|_K$, $d := \dim_H(K)$, such that $C^{-1}r^d \leq \mu(B(x,r)) \leq Cr^d$ for all $x \in K$, $r > 0$;
- there exists $\lambda > 1$ such that, for all $\rho > 0$, K can be covered by a collection $\mathcal{C}_\rho(K)$ of disjoint cubes with sizes belonging to the interval $[\rho, \lambda\rho]$ such that their mutual distances are at least $\lambda^{-1}\rho$ and each of them contain a ball of radius $\lambda^{-1}\rho$ about some point of K.

In the context of Theorem 1.2, recall that we are also assuming that

$$n \geq d > \max\{\sqrt{3}, \frac{(n-1) + \sqrt{(n-1)(n+3)}}{2}\}. \tag{3.10}$$

Furthermore, up to rescaling, we can suppose that $K \subset [0,1]^n$.

Let us now fix some rational parameters

$$\max\{\frac{3d+3}{d+3}, \frac{(n+1)d + (n-1)}{d+2}\} < n - \varepsilon_0 < s_0'' < s_0' < s_0 < d \leq n, \tag{3.11}$$

$$\alpha := \min\left\{s_0'' - 1, 2 - \frac{s_0' - (n-1)}{2}\right\}, \tag{3.12}$$

$$\frac{\varepsilon_0 d}{2} < \frac{\varepsilon_1}{2} < \min\{s_0'' - 1, 1\} - \frac{\varepsilon_0 d}{2} - 2\varepsilon_0 - d + n, \tag{3.13}$$

and

(3.14)

$$0 < \varepsilon_* < \min\left\{s_0' + \varepsilon_0 - n, 2\left(\min\{s_0'' - 1, 1\} - \frac{\varepsilon_0 d}{2} - 2\varepsilon_0 - d + n - \frac{\varepsilon_1}{2}\right)\right\}.$$

Note that these conditions are mutually compatible: indeed, our assumption (3.10) allows us to choose s_0, s_0', s_0'' and ε_0 in (3.11); since $\varepsilon_0 < \min\left\{\frac{3-d}{d+3}, \frac{n-d+1}{d+2}\right\}$ and $s_0' > n - \varepsilon_0$, we can select ε_1 in (2.4) and ε_* in (2.5).

In what follows, $\delta = 2^{-N}$, $N \in \mathbb{N}$, is an arbitrary (small) dyadic scale such that δ^{ε_0} is also a dyadic scale.

Our plan is to show Theorem 1.2 by following the same arguments from the previous section *after* some adjustments in the definitions and arguments.

3.2. **Absence of light cubes.** In comparison with the previous section, our current setting is technically easier because there are no δ-light cubes in the sense that any $Q \in \mathcal{C}_\delta(K)$ satisfies

$$\mu(Q) \geq C^{-1}\lambda^{-d}\delta^d =: c_{11}\delta^d. \tag{3.15}$$

3.3. **Exceptional directions.** Given a cube $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$, we define

$$E_{\delta,Q} := \left\{ e \in S^{n-1} : \int_{e^\perp} |\widehat{\mu_Q}(\zeta)|^2 \cdot |\zeta|^{s_0'-(n-1)}\, d\mathcal{H}^{n-1}(\zeta) \geq \delta^{-\varepsilon_1} \right\}$$

where $\mu_Q = \mu|_Q$. Since $s_0' < d$, we have that

$$\mathcal{H}^{n-1}(E_{\delta,Q}) \leq c_2(s_0', n) I_{s_0'}(\mu)\delta^{\varepsilon_1} \tag{3.16}$$

for all $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$.

3.4. **Good and bad lines.** Denote by $\mathcal{L}_e$ the space of lines parallel to $e \in S^{n-1}$. Given a cube $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$, the set $\mathcal{L}_{e,\delta,\text{bad},Q}$ of δ-*bad lines* in direction e associated to Q consists of all lines $\ell \in \mathcal{L}_e$ disjoint from $K \cap Q$ whose 2δ-neighborhood $\ell(2\delta)$ satisfy

$$\#\left\{R \in \mathcal{C}_\delta(K) : R \cap Q \neq \emptyset, R \cap \ell(2\delta) \neq \emptyset\right\} \geq \delta^{2\varepsilon_0 - 1}.$$

We say that $\ell \in \mathcal{L}_e$ is a δ-*good line* in the direction e whenever $\ell \notin \mathcal{L}_{e,\delta,\text{bad},Q}$ for all $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$. The collection of δ-good lines in the direction e is denoted by $\mathcal{L}_{e,\delta,\text{good}}$ and we define

$$L_{e,\delta,\text{good}} := \bigcup_{\ell \in \mathcal{L}_{e,\delta,\text{good}}} \ell.$$

Lemma 3.1. *$\mathcal{H}^{s_0'}_{\lambda\delta}(\text{Vis}_e(K) \cap L_{e,\delta,good}) \leq \delta^{\varepsilon_*}$ for all δ sufficiently small.*

Proof. The argument below is parallel to the proof of Lemma 2.5 above. Once again, let $\mathcal{T}_{e,\delta}$ be a collection of tubes of width δ whose bases are perpendicular to e in order to cover $[0,1]^n$, so that our task is reduced to prove that, for each $T \in \mathcal{T}_{e,\delta}$, the minimal number $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta)$ of balls of radii in the interval $[\delta, \lambda\delta]$ needed to cover $\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T$ is at most

$$N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq c_5(n)\delta^{\varepsilon_0 - 1}.$$

In order to estimate $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta)$ for a given $T \in \mathcal{T}_{e,\delta}$, we consider two scenarios:

(i) for all $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$, one has
$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q \neq \emptyset, R \cap T \neq \emptyset\} < \delta^{2\varepsilon_0 - 1};$$

(ii) there exists $Q_1 \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ with
$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q_1 \neq \emptyset, R \cap T \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

In the first scenario, we have that $N(\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T, \delta) \leq \delta^{\varepsilon_0 - 1}$ simply because T can meet at most $\delta^{-\varepsilon_0}$ cubes $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$.

In the second scenario, we take Q_1 to be a $\leq_e$-minimal cube with the property described in (ii) (in the sense that Q_1 minimizes $\inf\{\langle x, e\rangle : x \in Q_1\}$ among all cubes in (ii)). Since the 2δ-neighborhood of any line $\ell \subset T$ contains T, we also have

$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q_1, R \cap \ell(2\delta) \neq \emptyset\} \geq \delta^{2\varepsilon_0 - 1}.$$

Therefore, it follows from the definition of δ-good line that any $\ell \in \mathcal{L}_{e,\delta,\text{good}}$ included in T must intersect $K \cap Q_1$.

We affirm that

$$\text{Vis}_e(K) \cap L_{e,\delta,\text{good}} \cap T \cap Q = \emptyset$$

for any cube $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ with $\inf\{\langle x, e\rangle : x \in Q\} > \sup\{\langle y, e\rangle : y \in Q_1\}$. In fact, if $x \in \mathrm{Vis}_e(K) \cap L_{e,\delta,\mathrm{good}} \cap T \cap Q$, then $\pi_e(x) = \pi_e(y)$ for some $y \in Q_1$. Since $\langle x, e\rangle > \langle y, e\rangle$, one would get $x \notin \mathrm{Vis}_e(K)$, a contradiction.

Hence, $\mathrm{Vis}_e(K) \cap L_{e,\delta,\mathrm{good}} \cap T$ is covered by the collection of cubes $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ with $\inf\{\langle x, e\rangle : x \in Q\} \le \sup\{\langle y, e\rangle : y \in Q_1\}$. Now, we observe that

- the number of cubes $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ intersecting T with
$$\inf\{\langle z, e\rangle : z \in Q_1\} \le \inf\{\langle x, e\rangle : x \in Q\} \le \sup\{\langle y, e\rangle : y \in Q_1\}$$
is bounded by an absolute constant $c_4(n)$; for each of them, we will use the crude bound $N(\mathrm{Vis}_e(K) \cap L_{e,\delta,\mathrm{good}} \cap T, \delta) \le \delta^{\varepsilon_0 - 1}$ coming from the fact that $Q \cap T$ can be covered using at most δ^{ε_0-1} balls of radius δ;
- any cube $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ intersecting $T \cap K$ with
$$\inf\{\langle z, e\rangle : z \in Q_1\} > \inf\{\langle x, e\rangle : x \in Q\}$$
satisfies
$$\#\{R \in \mathcal{C}_\delta(K) : R \cap Q_1 \neq \emptyset, R \cap \ell(2\delta) \neq \emptyset\} \le \delta^{2\varepsilon_0 - 1}$$
because of the $\le_e$-minimality of Q_1; the number of such cubes Q is at most $\le \delta^{-\varepsilon_0}$ because T meets at most $\delta^{-\varepsilon_0}$ cubes $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$.

By combining the estimates above, we conclude that
$$N(\mathrm{Vis}_e(K) \cap L_{e,\delta,\mathrm{good}} \cap T, \delta) \le c_4(n)\delta^{\varepsilon_0-1} + \delta^{-\varepsilon_0}\delta^{2\varepsilon_0-1} = c_5(n)\delta^{\varepsilon_0-1}.$$
This completes the proof. □

3.5. **Typical visible parts in bad lines.** Similarly to the previous section, the last step towards the proof of Theorem 1.2 is the following estimate:

Lemma 3.2. *Let $Q \in \mathcal{C}_{\delta^{\varepsilon_0}}(K)$ be a cube, consider a direction $e \notin E_{\delta,Q}$, and denote $L_{e,\delta,bad,Q} := \bigcup_{\ell \in \mathcal{L}_{e,\delta,bad,Q}} \ell$. Then,*
$$\mathcal{H}^{s_0'-1}_\infty(\pi_e(L_{e,\delta,bad,Q})) \le \delta^{\varepsilon_* + \varepsilon_0 d}$$
for all δ sufficiently small.

Proof. By contradiction, suppose that $\mathcal{H}^{s_0'-1}_\infty(\pi_e(L_{e,\delta,\mathrm{bad},Q})) \ge \delta^{\varepsilon_*+\varepsilon_0 d}$. By Lemma 2.1, we have a probability measure ν supported on $H_{e,\delta,Q} := \pi_e(L_{e,\delta,\mathrm{bad},Q})$ with
$$\nu(B(x,r)) \le C(n-1)\delta^{-\varepsilon_*-\varepsilon_0 d} r^{s_0'-1}$$
for all $x \in H$ and $r > 0$. Thus, our choice of $\alpha \le s_0'' - 1 < s_0' - 1$ in (3.12) (and Remark 2.2) means that the α-energy of ν satisfies
$$c_1(\alpha, n-1)\int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^\alpha \, d\xi = I_\alpha(\nu) \le c_6(s_0'', s_0', n)\delta^{-\varepsilon_*-\varepsilon_0 d}. \tag{3.17}$$

Next, we observe that, by definition, any line $\ell \in \mathcal{L}_{e,\delta,\mathrm{bad},Q}$ misses $K \cap Q$. Therefore, $\mu_{Q,e} := (\pi_e)_*(\mu_Q)$ and ν have disjoint supports. Hence, if we fix a non-negative smooth bump function φ on $e^\perp \simeq \mathbb{R}^{n-1}$ with total integral one and $\varphi(0) = 1$, then
$$\begin{aligned} 0 &= \int \varphi_\eta * \mu_{Q,e}\, d\nu = \int \widehat{\varphi}(\eta\xi)\widehat{\mu_{Q,e}}(\xi)\overline{\widehat{\nu}(\xi)}\, d\xi \\ &= \int (1 - \widehat{\varphi}(c_7(n)\delta\xi))\widehat{\varphi}(\eta\xi)\widehat{\mu_{Q,e}}(\xi)\overline{\widehat{\nu}(\xi)}\, d\xi + \int \widehat{\varphi}(c_7(n)\delta\xi)\widehat{\varphi}(\eta\xi)\widehat{\mu_{Q,e}}(\xi)\overline{\widehat{\nu}(\xi)}\, d\xi \\ &:= A_2 - A_1 \end{aligned}$$
for all $0 < \eta \ll \delta$, where $\varphi_\eta(x) = \varphi(\eta x)/\eta^{n-1}$.

Once more, we will reach a contradiction with the identity in the previous paragraph by showing that $|A_2| < |A_1|$. For this sake, we observe that $\widehat{\varphi}$ is a bounded Lipschitz function with $\widehat{\varphi}(0) = 1$, so that $|1 - \widehat{\varphi}(c_7(n)\delta\xi)| \leq c_8(n)\delta|\xi|$ and, *a fortiori*,

$$|A_2| \leq c_8(n)\delta^{\frac{s_0'-(n-1)}{2}+\frac{\alpha}{2}} \left(\int |\widehat{\mu_{Q,e}}(\xi)|^2 \cdot |\xi|^{s_0'-(n-1)}\, d\xi\right)^{1/2} \left(\int |\widehat{\nu}(\xi)|^2 \cdot |\xi|^{\alpha}\, d\xi\right)^{1/2}$$

thanks to our choice of $\frac{s_0'-(n-1)}{2} + \frac{\alpha}{2} \leq 1$ in (3.12) and the Cauchy–Schwarz inequality. By plugging into the previous inequality the facts that our choices in (3.11) and (3.12) imply $\frac{s_0'-(n-1)}{2} + \frac{\alpha}{2} \geq \min\{s_0'' - 1, 1\}$, our assumption $e \notin E_{\delta,Q}$ allows (by definition) to control $|\widehat{\mu_Q}(\xi)|$ $(= |\widehat{\mu_{Q,e}}(\xi)|$ for $\xi \in e^{\perp})$, and the α-energy of ν is controlled by (3.17), we derive that

$$A_2 \leq c_9(s_0'', s_0', n)\delta^{\min\{s_0''-1,1\}}\delta^{-\varepsilon_1/2}\delta^{-(\varepsilon_*+\varepsilon_0 d)/2}.$$

On the other hand, if we write

$$A_1 = \int \varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r)\, d\nu(r),$$

and we recall that ν is supported in $H_{e,\delta,Q} := \pi_e(L_{e,\delta,\text{bad},Q})$, then we can use the fact that $r \in H_{e,\delta,Q}$ means $\ell := \pi_e^{-1}(r) \in \mathcal{L}_{e,\delta,\text{bad},Q}$, i.e., $\ell(2\delta)$ meets at least $\delta^{2\varepsilon_0-1}$ cubes $R \in \mathcal{C}_\delta(K)$ intersecting Q and verifying (3.15), to deduce that $\mu_Q(\ell(2\delta)) \geq c_{11}\delta^{2\varepsilon_0-1+d}$ and, *a fortiori*,

$$\varphi_{c_7(n)\delta} * \varphi_\eta * \mu_{Q,e}(r) \geq c_{11}c_{10}(n)\delta^{2\varepsilon_0-1+d-(n-1)}$$

for all $r \in H$ and $0 < \eta \ll \delta$. Therefore,

$$A_1 \geq c_{11}c_{10}(n)\delta^{2\varepsilon_0+d-n}$$

because ν is a probability measure on H.

At this point, we get the desired contradiction $A_1 > |A_2|$ for δ is sufficiently small because our choice (3.14) implies that $2\varepsilon_0 + d - n < \min\{s_0''-1, 1\} - \frac{\varepsilon_1}{2} - \frac{\varepsilon_*+\varepsilon_0 d}{2}$. □

3.6. **End of the proof of Theorem 1.2.** Let us take a decreasing sequence of dyadic scales $\delta_j \to 0$ such that $\delta_j^{\varepsilon_0}$ also a dyadic scale. We define the set E_{δ_j} of δ_j-exceptional directions as

$$E_{\delta_j} := \bigcup_{Q \in \mathcal{C}_{\delta_j}(K)} E_{\delta_j,Q}.$$

Since $\#\mathcal{C}_\eta(K) \leq c_{12}\eta^{-d}$ (thanks to (3.15) and the finiteness of μ), it follows from (3.16) that

$$\mathcal{H}^{n-1}(E_{\delta_j}) \leq c_2(s_0', n) I_{s_0'}(\mu)\delta_j^{\varepsilon_1-\varepsilon_0 d}.$$

Therefore, our choice of $\varepsilon_1 > \varepsilon_0 d$ in (3.13) implies

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(E_{\delta_j}) < \infty,$$

so that the set

$$E = E(s_0, s_0', s_0'', \varepsilon_0, \varepsilon_1, \varepsilon_*) := \bigcap_{n=1}^{\infty} \bigcup_{j \geq n} E_{\delta_j}$$

has zero $\mathcal{H}^{n-1}$-measure.

We affirm that $\dim_H(\text{Vis}_e(K)) \leq s_0'$ whenever $e \in S^{n-1} \setminus E$. In fact, an element $e \notin E$ belongs to finitely many E_{δ_j}'s, say $e \notin E_{\delta_j}$ for all $j \geq j_e$.

By Lemma 3.1, $\mathcal{H}^{s_0'}_{\lambda\delta_j}(\text{Vis}_e(K)\cap L_{e,\delta_j,\text{good}}) \leq \delta_j^{\varepsilon_*}$ for all j sufficiently large. Moreover, $\mathcal{H}^{s_0'}_{\infty}\left(\text{Vis}_e(K)\cap \bigcup\limits_{Q\in\mathcal{C}_{\delta_j^{\varepsilon_0}}(K)} L_{e,\delta_j,\text{bad},Q}\right) \leq c_{12}\delta_j^{\varepsilon_*}$ for all $j \geq j_e$ sufficiently large by Lemma 3.2 (and the fact that $\#\mathcal{C}_{\delta_j^{\varepsilon_0}}(K) \leq c_{12}\delta_j^{-\varepsilon_0 d}$).

By putting these three estimates together, we derive that if $e \notin E$, then

$$\mathcal{H}^{s_0'}_{\infty}(\text{Vis}_e(K)) \leq (c_{12}+1)\delta_j^{\varepsilon_*}$$

for all $j \geq j_e$ sufficiently large, and, consequently, $\dim_H(\text{Vis}_e(K)) \leq s_0'$ for all $e \notin E(s_0, s_0', s_0'', \varepsilon_0, \varepsilon_1, \varepsilon_*)$.

Since $s_0, s_0', s_0'', \varepsilon_0, \varepsilon_1, \varepsilon_*$ are arbitrary rational parameters satisfying (3.11), (3.12), (3.13) and (3.14), we conclude that

$$\dim_H(\text{Vis}_e(K)) \leq \max\{\frac{3d+3}{d+3}, \frac{(n+1)d+(n-1)}{d+2}\}$$

for Lebesgue almost every $e \in S^{n-1}$.

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