THE BEGINNING OF THE LAGRANGE SPECTRUM OF CERTAIN ORIGAMIS OF GENUS TWO

CARLOS MATHEUS

Abstract. The initial portion of the Lagrange spectrum L_{B7} of certain square-tiled surfaces of genus two was described in details in the work of Hubert–Lelièvre–Marchese–Ulcigrai. In particular, they proved that the smallest element of L_{B7} is an isolated point ϕ_1, but the second smallest value ϕ_2 of L_{B7} is an accumulation point. Also, they conjectured that the portion $L_{B7} \cap [\phi_2, \eta_1)$ is a Cantor set for a specific value η_1 and they asked about the continuity properties of the Hausdorff dimension of $L_{B7} \cap (-\infty, t)$ as a function of $t < \eta_1$. In this note, we solve affirmatively these problems.

1. Introduction

The classical Lagrange spectrum L was originally introduced in relation to the study of Diophantine approximations of irrational numbers and, alternatively, it can also be seen as the set of real numbers encoding cusp excursions of geodesics on the modular surface, i.e.,

$$L = \left\{ \limsup_{t \to \infty} \frac{2}{\mathrm{sys}(g_t(X))^2} < \infty : X \in \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z}) \right\},$$

where $g_t := \mathrm{diag}(e^t, e^{-t})$ and $\mathrm{sys}(Y) := \min\{|h(v)|_{\mathbb{R}^2} : v \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$ for $Y = h \cdot \mathrm{SL}(2,\mathbb{Z}) \in \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$.

This point of view led Hubert–Marchese–Ulcigrai [5] to naturally extend the notion of Lagrange spectrum to the context of *Teichmüller dynamics* (see, e.g., Zorich's survey [8] for the basic aspects of this theory).

More concretely, they defined the Lagrange spectrum $L_{\mathcal{I}}$ associated to the closure $\mathcal{I}$ of a $\mathrm{SL}(2,\mathbb{R})$-orbit on the moduli space of unit area translation surfaces as

$$L_{\mathcal{I}} = \left\{ \limsup_{t \to \infty} \frac{2}{\mathrm{sys}(g_t(X))^2} < \infty : X \in \mathcal{I} \right\},$$

where the action of g_t is the so-called Teichmüller geodesic flow and $\mathrm{sys}(Y)$ is the minimal length of a saddle-connection of Y. Also, they showed that $L_{\mathcal{I}}$ shares some common features with the classical Lagrange spectrum, e.g.,

- if $\mathcal{I}$ consists of some translation surfaces with genus g and σ conical singularities, then $L_{\mathcal{I}}$ is a subset of $[\frac{(2g-2+\sigma)\pi}{2}, \infty)$ given by the closure of the maximal values of the function $Y \mapsto \frac{2}{\mathrm{sys}(Y)^2}$ along g_t-periodic orbits included in $\mathcal{I}$;
- if $\mathcal{I}$ contains a square-tiled surface, then $L_{\mathcal{I}}$ contains a Hall's ray, i.e., $[r, \infty) \subset L_{\mathcal{I}}$ for some $r > 0$.

On the other hand, it was discovered by Hubert–Lelièvre–Marchese–Ulcigrai [4] that the beginning of the Lagrange spectra of $\mathrm{SL}(2,\mathbb{R})$-orbits of square-tiled surfaces might behave differently from the classical Lagrange spectrum. More precisely, let X be the square-tiled surface of genus two with unit area obtained from seven squares $sq(k)$, $1 \leq k \leq 7$, in $\mathbb{R}^2$ with areas $1/7$ by gluing the right vertical side of

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$sq(k)$ to the left vertical side of $sq(h(k))$ and the top horizontal side of $sq(k)$ to the bottom horizontal side of $sq(v(k))$, where h and v are the permutations with cycles

$$h = (1,2,3)(4)(5)(6)(7) \quad \text{and} \quad v = (1,4,5,6,7)(2)(3).$$

By following the terminology of Hubert–Lelièvre [3], the $\mathrm{SL}(2,\mathbb{R})$-orbit of X is called $B7$. It was shown by Hubert–Lelièvre–Marchese–Ulcigrai that the Lagrange spectrum L_{B7} associated to $B7$ starts with an isolated point and an accumulation point[1], namely:

$$[0, \phi_2) \cap L_{B7} = \{\phi_1\} \quad \text{and} \quad \phi_2 \in L'_{B7},$$

where[2] $\phi_1 := 7+14\cdot[0;\overline{3,1}] = 10.692676\dots$ and $\phi_2 := 14\cdot[0;1,4,\overline{1,3}] = 11.582575\dots$. Furthermore, they conjectured that

$$\mathbb{K} := L_{B7} \cap [\phi_2, \eta_1)$$

is a Cantor set and they asked whether the Hausdorff dimension of $L_{B7} \cap (-\infty, t)$ varies continuously with $\phi_1 < t < \eta_1 := 7 \cdot \frac{[5;1,4,2,\overline{1,5}]+[0;1,5,1,\overline{1,5}]}{4} = 11.655309\dots$.

In this note, we show that:

Theorem 1.1. *The Hausdorff dimension $d(t)$ of $L_{B7} \cap (-\infty, t)$ varies continuously with $t < \eta_1$.*

Theorem 1.2. *The portion $\mathbb{K} = L_{B7} \cap [\phi_2, \eta_1)$ of L_{B7} is a Cantor set.*

Remark 1.3. *We will also show that $0.30944 < d(\eta_1) < 0.30976$, any $\phi \in \mathbb{K}$ is accumulated by Cantor sets with positive Hausdorff dimensions contained in $\mathbb{K}$, and $d(t)$ is not Hölder continuous.*

For the sake of exposition, we divide the rest of this note into five sections: first, we review some results from [4] about the description of the initial portion of L_{B7}; next, we employ the results of Cerqueira, Moreira and the author [2] to deduce the continuity of the Hausdorff dimension of $L_{B7} \cap (-\infty, t)$ as a function of $t \in (-\infty, \eta_1)$; afterwards, we show that $\mathbb{K}$ is a Cantor set; then, we modify an argument of Moreira [7] in order to prove that any $\phi \in \mathbb{K}$ is accumulated by Cantor sets with positive Hausdorff dimensions contained in $\mathbb{K}$; finally, we show that $d(t)$ is not Hölder continuous near ϕ_2.

2. Preliminaries

Consider the left shift dynamics $\sigma : \{a,b\}^{\mathbb{Z}} \to \{a,b\}^{\mathbb{Z}}$ on the symbolic space $\Sigma := \{a,b\}^{\mathbb{Z}}$ where $a := 1,4,2,4$ and $b := 1,3$. It was shown in [4, §4.5] that

$$L_{B7} \cap (-\infty, \eta_1) = \{\phi_1\} \cup \mathbb{K} = \left\{ L^\sigma(\xi) := \limsup_{n\to\infty} h(\sigma^n(\xi)) : \xi \in \Sigma \right\}$$

where $h : \Sigma \to \mathbb{R}$ is the height function given by

$$h((\xi_n)_{n\in\mathbb{Z}}) := \begin{cases} 7\cdot([0;1,4,\xi_1,\xi_2,\dots] + [0;1,4,\xi_{-1},\xi_{-2},\dots]), & \text{if } \xi_0 = a, \\ 7\cdot(1 + [0;3,\xi_1,\xi_2,\dots] + [0;3,\xi_{-1},\xi_{-2},\dots]), & \text{if } \xi_0 = b. \end{cases}$$

Also, it is essentially proved in [4, §5] that $\mathbb{K}$ is a perfect set. Indeed, if $\phi \in \mathbb{K} \setminus \{\phi_2, \phi_\infty\}$ with $\phi_\infty := 14\cdot[0,1,4,\overline{1,4,2,4}] = h(\overline{a})$, then Lemmas 5.1, 5.2 and 5.4 of their article give one of the following two scenarios:

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[1] This contrasts with the classical Lagrange spectrum because $L \cap (-\infty, 3) = \{k_1 < k_2 < \dots < k_n < \dots\}$ where k_n is an explicit increasing sequence converging to 3.

[2] We are using the notations $[a_0; a_1, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots}}$ and $\overline{c_1,\dots,c_k} = c_1,\dots,c_k,c_1,\dots c_k,\dots$

(i) there exists $k \geq 2$ such that

$$\phi = \begin{cases} 14 \cdot [0; 1, 4, a^{(i)}, \overline{b}], & \text{if } k = 2i+1 \text{ is odd,} \\ 7 \cdot ([0; 1, 4, a^{(i+1)}, \overline{b}] + [0; 1, 4, a^{(i)}, \overline{b}]), & \text{if } k = 2i+2 \text{ is even,} \end{cases}$$

where $c^{(j)} := \underbrace{c \dots c}_{j \text{ times}}$, or

(ii) there are $k, n \geq 1$ such that $\phi = L^\sigma(\xi)$ where $\xi \in \Sigma$ contains infinitely many copies of $a^{(k)} b^{(n)} a$ but no copies of $a^{(k+1)}$ and no copies of $a^{(k)} b^{(n-1)} a$.

By applying again Lemmas 5.1, 5.2 and 5.4 of their article, we have that $\phi \in \mathbb{K}'$ because

- in the first case, $\phi = \lim\limits_{m \to \infty} L^\sigma(\overline{a^{(k)} b^{(m)}})$, and
- in the second case, $\phi = \lim\limits_{m \to \infty} L^\sigma(\overline{b^{(P(m))} \xi_{Q(m)+1} \dots \xi_{R(m)-1} b^{(S(m))}})$, where $\xi_{Q(m)}$ and $\xi_{R(m)}$ correspond to the last letter a in appropriately chosen occurrences of the block $a^{(k)} b^{(n)} a$ in ξ, and $P(m)$ and $S(m)$ are suitably large in comparison $P(m-1)$, $Q(m-1)$, $R(m-1)$ and $S(m-1)$.

Since Theorem 1.1 of their article ensures that $\phi_2, \phi_\infty \in \mathbb{K}'$, we have that $\mathbb{K}$ is a closed set without no isolated points.

3. Proof of Theorem 1.1

It is well-known [1] that the left-shift dynamics on $\{1, 2, 3, 4\}^{\mathbb{Z}}$ can be smoothly realized via the natural extension $\varphi(x, y) = (\{1/x\}, 1/(\lfloor 1/x \rfloor + y))$ of the Gauss map $g([0; a_1, a_2, \dots]) := [0; a_2, \dots]$. Since φ is a smooth area-preserving diffeomorphism whose local stable and unstable manifolds are parallel to the axes and the gradient of the smooth realization of the height function h is transverse to the axes, the key results from [2] can be employed to derive that:

- the Hausdorff dimension $d(t)$ of $\{L^\sigma(\xi) : \xi \in \Sigma\} \cap (-\infty, t)$ depends continuously on $t \in \mathbb{R}$, and
- $d(\eta_1) = 2 \cdot D(\eta_1)$, where $D(\eta_1)$ is the Hausdorff dimension of Cantor set $C(a, b)$ of real numbers with continued fraction expansions in $\Sigma^+ = \{a, b\}^{\mathbb{N}}$.

At this point, the desired theorem follows from the fact that $L_{B7} \cap (-\infty, t) = \{L^\sigma(\xi) : \xi \in \Sigma\} \cap (-\infty, t)$ for all $t < \eta_1$.

4. Proof of Theorem 1.2

We saw in Section 2 that $\mathbb{K}$ is a perfect set. Therefore, our task of showing that $\mathbb{K}$ is a Cantor set can be reduced to prove that $d(\eta_1) = 2 \cdot D(\eta_1) < 1$.

In the sequel, we will show that $D(\eta_1) = 0.154\dots$. For this sake, we observe that

$$C(a, b) = \bigcap_{n \in \mathbb{N}} \psi^{-n}(I_b \cup I_a)$$

where $I_b = [[0; \overline{b}], [0; b\overline{a}]]$, $I_a = [[0; a\overline{b}], [0; \overline{a}]]$, and $\psi : I_b \cup I_a \to [[0; \overline{b}], [0; \overline{a}]]$, $\psi|_{I_b}(x) = g^2(x)$, $\psi|_{I_a}(x) = g^4(x)$. Hence, we can use the method described in [6, §4] to obtain that, for all $n \in \mathbb{N}$, one has

$$\alpha_n \leq D(\eta_1) \leq \beta_n$$

where

$$\sum_{(x_1, \dots, x_k) \in \{a, b\}^n} \left(\min \left\{ \prod_{i=1}^{k} [0; x_i, \dots, x_k, \overline{1, 3}], \prod_{i=1}^{k} [0; x_i, \dots, x_k, \overline{1, 4, 1, 2}] \right\} \right)^{2\alpha_n} = 1$$

and

$$\sum_{(x_1,\dots,x_k)\in\{a,b\}^n}\left(\max\left\{\prod_{i=1}^{k}[0;x_i,\dots,x_k,\overline{1,3}],\prod_{i=1}^{k}[0;x_i,\dots,x_k,\overline{1,4,1,2}]\right\}\right)^{2\beta_n}=1.$$

A quick computer search for the values of α_4 and β_4 shows that

$$0.15472<\alpha_4\leq D(\eta_1)\leq\beta_4<0.15488.$$

5. Local structure of $\mathbb{K}$

Recall that $\mathbb{K}$ is a Cantor set. In particular, any $x\in\mathbb{K}$ is accumulated by a sequence $x_n\in\mathbb{K}$ with $x_n\neq x$. In what follows, we adapt the proof of Theorem 3 in [7] to show that x is accumulated by Cantor sets of positive Hausdorff dimensions included in $\mathbb{K}$.

In this direction, let us take $\xi^{(n)}=(\xi_j^{(n)})_{j\in\mathbb{Z}}\in\Sigma$ such that $x_n=L^\sigma(\xi^{(n)})$. We have $x_n=7\cdot\limsup_{j\to\infty}([0;1,4,\xi_{j+1}^{(n)},\xi_{j+2}^{(n)},\dots]+[0;1,4,\xi_{j-1}^{(n)},\xi_{j-2}^{(n)},\dots])$.

Given $\delta>0$, we can fix $n_0\in\mathbb{N}$ large such that, for each $n\geq n_0$, one has $|L^\sigma(\xi^{(n)})-x|<\delta$ and $|7([0;1,4,\xi_{j+1}^{(n)},\xi_{j+2}^{(n)},\dots]+[0;1,4,\xi_{j-1}^{(n)},\xi_{j-2}^{(n)},\dots])-x|<\delta$ for infinitely many $j\in\mathbb{N}$.

Let $N=\lceil\delta^{-1}\rceil$ and, for each j and n as above, consider the finite sequence with $2N+1$ terms $(\xi_{j-N}^{(n)},\dots,\xi_j^{(n)},\dots,\xi_{j+N}^{(n)})=:S(j,n)$. By the pigeonhole principle, there exists a finite string S such that, for infinitely many values of n, the string S appears infinitely many times as $S(j,n)$, i.e., there is an infinite set $A\subset\mathbb{N}$ so that for each $n\in A$ we can find $j_1(n)<j_2(n)<\dots$ with $\lim\limits_{i\to\infty}(j_{i+1}(n)-j_i(n))=\infty$ and $S(j_i(n),n)=S$ for all $i\geq 1$.

Consider the sequences $\beta(i,n)$ for $i\geq1$, $n\in A$ given by

$$\beta(i,n)=(\xi_{j_i(n)+N+1}^{(n)},\xi_{j_i(n)+N+2}^{(n)},\dots,\xi_{j_{i+1}(n)+N}^{(n)}).$$

Since the sequence $(x_n)_{n\in A}$ is not constant, there are (i_1,n_1) and (i_2,n_2) so that $\beta(i_1,n_1)$ and $\beta(i_2,n_2)$ can not be expressed as concatenations of copies of some finite string γ. This implies that $B=\{\beta(i_1,n_1)\beta(i_2,n_2),\beta(i_2,n_2)\beta(i_1,n_1)\}^{\mathbb{Z}}$ is a Bernoulli subshift of Σ such that $\{L^\sigma(\beta):\beta\in B\}$ is a portion of $\mathbb{K}$ included in the (2δ)-neighborhood of x. By Proposition 2.16 of [2], $\{L^\sigma(\beta:\beta\in B)\}$ contains a Cantor set of positive Hausdorff dimension, so that the argument is complete.

6. Local dimension of $\mathbb{K}$ near ϕ_2

The Hausdorff dimension $d(t)$ of $L_{B7}\cap(-\infty,t)$ is not α-Hölder continuous near ϕ_2: otherwise, the restriction of d to $L_{B7}\cap[\phi_2,\phi_2+\varepsilon]$ would be a α-Hölder continuous function from a set of Hausdorff dimension $d(\phi_2+\varepsilon)$ to the interval $[0,d(\phi_2+\varepsilon)]$; since this interval is non-trivial when $\varepsilon>0$ (thanks to the result from the previous section), its Hausdorff dimension is 1 and, *a fortiori*, $d(\phi_2+\varepsilon)\geq\alpha$ for all $\varepsilon>0$, a contradiction with the continuity of d at the point ϕ_2 (where $d(\phi_2)=0$).

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Carlos Matheus

CMLS, CNRS, École polytechnique, Institut Polytechnique de Paris, 91128 Palaiseau Cedex, France.

E-mail address: `carlos.matheus@math.cnrs.fr`