PALIS' CONJECTURES: A QUEST FOR A GLOBAL QUALITATIVE VIEW OF MOST DYNAMICAL SYSTEMS

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To Jacob Palis on the occasion of his 80th anniversary

ABSTRACT. About 25 years ago, Jacob Palis proposed his *global conjecture*, i.e., an ambitious collection of five statements aiming to describe the asymptotic behaviors of the majority of the orbits for a dense set of dynamical systems. Furthermore, Jacob Palis formulated more conjectures intended to pave a way to solve the global conjecture: in a nutshell, he suggests to focus on establishing his global conjecture near two types of bifurcations called *homoclinic tangencies* and *heterodimensional cycles* because he predicts that these two bifurcations can be densely found outside the well-understood realm of uniformly hyperbolic systems.

In this survey article, we review the historical context behind some of Palis' conjectures and, after that, we briefly discuss some landmarking results towards these conjectures.

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I'm grateful to Laure Saint-Raymond for her kind invitation to write this survey text near the 25th anniversary of Palis' conjectures and the 80th birthday of J. Palis. I learned many of the topics discussed in this text from several conversations with some of my coauthors – including C. G. Moreira, J. Palis and J.-C. Yoccoz – and I'm grateful to them for generously sharing with me their visions on Palis' conjectures.

1. Brief prehistory of Palis' conjectures

1.1. **Poincaré's "méthodes nouvelles de la mécanique celeste".** The revolutionary works of Henri Poincaré (circa 1890) in relation to the fundamental question of the stability of the solar system in celestial mechanics famously introduced a huge amount of new methods for the study of ordinary differential equations. For instance:

- Poincaré advocated in [42] that one should seek for *qualitative* properties of general ordinary differential equations because, despite its usefulness in some *practical* applications, the *quantitative* techniques describing the solutions of the problems in celestial mechanics usually lead to *divergent* series expansions[1];
- Poincaré said in [41] that the notion of *homoclinic orbits* deserves a special attention in the qualitative theory of ordinary differential equations because

  "... *rien n'est plus propre à nous donner une idée de la complication de tous les problèmes de dynamique* ..."
- etc.

The modern definition of homoclinic orbit of a diffeomorphism $f : M \to M$ of a compact manifold M goes as follows. Recall that a periodic point $p \in M$ of f is a fixed point $f^n(p) = p$ of some iterate $f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}}$ of f. Moreover, the minimal period k of a periodic point p is the smallest integer $k \in \mathbb{N}$ with $f^k(p) = p$. In this setting, the orbit $\{f^n(q) : n \in \mathbb{Z}\}$ of a point $q \neq p$ is *homoclinic* to a periodic point p of minimal period k whenever $f^{nk}(q) \to p$ as $n \to \pm\infty$, that is, the orbit of q is accumulates the orbit of p both in the past and the future.

In 1935, George Birkhoff confirmed Poincaré's foresight about dynamical complications associated to homoclinic orbits by showing that a "generic" homoclinic orbit is accumulated by periodic orbits of *arbitrarily high* (minimal) period: in particular, the presence of a "generic" homoclinic orbit forces the existence of *infinitely* many periodic points!

In subsequent years, Cartwright and Littlewood [14], [21] and [22], and Levinson [20] did extensive studies of homoclinic orbits in certain ordinary differential equations coming from Engineering problems. The intrinsic difficulty in the hard *analytical* approaches of Cartwright–Littlewood and Levinson motivated Steve Smale to try to find a soft *geometrical* explanation for the dynamical complexity steaming from "generic" homoclinic orbits.

After deeply thinking about this question on the beaches of Rio de Janeiro [46], Steve Smale found in 1967 the key geometric mechanism behind the results of Poincaré, Birkhoff, Cartwright–Littlewood, and Levinson: in a nutshell, the dynamical complexity observed near a general homoclinic orbit is due to the features of the so-called *Smale's horseshoe.*

[1] It is worth to notice that the article [42] has an incredible history: its first version won in 1889 a prize created by king Oscar of Sweden, but the text was completely rewritten before its publication in 1890 after L. Phragmén detected some serious difficulties in the portion of the initial version implicitly related to homoclinic orbits (cf. [5] and [48] resp. for excellent accounts in English and French resp. of this fruitful mistake of Poincaré).

1.2. **Smale's horseshoes associated to transverse homoclinic orbits.** Let $f : M \to M$ be a diffeomorphism possessing a *hyperbolic*[2] periodic point $p \in M$ of period $k \in \mathbb{N}$.

By the *stable manifold theorem*[3], the stable and unstable sets

$$W^s(p) := \{q \in M : f^{nk}(q) \to p \text{ as } n \to +\infty\} \quad \text{and} \quad W^u(p) := \{q \in M : f^{nk}(q) \to p \text{ as } n \to -\infty\}$$

of p are injectively *immersed* submanifolds of M. In this context, a *transverse homoclinic orbit* to p is the orbit of a point $q \in (W^s(p) \cap W^u(p)) \setminus \{p\}$ such that $T_qM = T_qW^s(p) \oplus T_qW^u(p)$.

The basic shape found by S. Smale near transverse homoclinic orbits is:

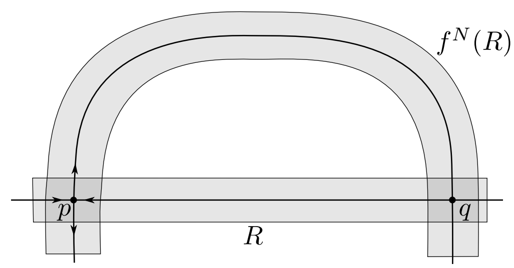

FIGURE 1. Smale's horseshoe.

In a few words, this picture is saying that a transverse homoclinic point q to a periodic point p gives rise to a rectangle R which is mapped by some appropriate iterate $F = f^N$ into a region $f^N(R)$ resembling a horseshoe.

From this figure, S. Smale inferred that the *maximal invariant set* $\Lambda := \bigcap\limits_{n\in\mathbb{Z}} F^n(R)$ consisting of all F-orbits never escaping R is a *hyperbolic set* of F in the sense of the following definition:

Definition 1. A compact g-invariant[4] subset $H \subset M$ is called a *hyperbolic set* of a diffeomorphism $g : M \to M$ if there are some constants $C > 0$, $0 < \lambda < 1$ and a splitting $T_xM = E^s(x) \oplus E^u(x)$ for each $x \in H$ such that:

- *the splitting is equivariant*: $dg(x) \cdot E^s(x) = E^s(g(x))$ and $dg(E^u(x)) = E^u(g(x))$;
- *the subbundles of the splitting are asymptotically contracted*[5]: $\|dg^n(x) \cdot v^s\| \leq C\lambda^n\|v^s\|$ and $\|dg^{-n}(x) \cdot v^u\| \leq C\lambda^n\|v^u\|$ for all $n \geq 0$, $v^s \in E^s(x)$ and $v^u \in E^u(x)$.

Moreover, S. Smale exploited the hyperbolic structure of the maximal invariant set $\Lambda := \bigcap\limits_{n\in\mathbb{Z}} F^n(R)$ to prove that $F|_\Lambda$ is a Markov process in disguise: if S_0 and S_1 are the connected components of $R \cap F(R)$, then the map $h : \Lambda \to \{0,1\}^{\mathbb{Z}} := \Sigma$ associating to $x \in \Lambda$ the symbolic itinerary $h(x) := (a_i)_{i\in\mathbb{Z}}$ of its F-orbit[6] turns out to be a topological conjugacy between $F|_\Lambda$ and

[2] I.e., the spectrum of the differential $df^k(p) : T_pM \to T_pM$ doesn't intersect the unit circle.

[3] See Appendix 1 of [37] for instance.

[4] That is, $g(H) = H$.

[5] Here, $\|.\|$ stands for a norm associated to some choice of Riemannian metric on M.

[6] I.e., $a_i \in \{0,1\}$ is defined by $F^i(x) \in S_{a_i}$

a Bernoulli shift on two symbols in the sense that h is a homeomorphism with $h(F(x)) = \sigma(h(x))$ where $\sigma : \Sigma \to \Sigma$ is the shift dynamics given by $\sigma((a_i)_{i\in\mathbb{Z}}) = (a_{i+1})_{i\in\mathbb{Z}}$.

In particular, this allowed S. Smale to recover G. Birkhoff's theorem (mentioned in Subsection 1.1 above) that the transverse homoclinic orbit q is accumulated by periodic orbits of arbitrarily high periods: indeed, this happens simply because $F|_\Lambda$ is topologically conjugated to σ and the set of periodic orbits of σ is easily shown to be dense in Σ !

1.3. **Uniformly hyperbolic diffeomorphisms.** The successful analysis of Smale's horseshoes was vastly extended by several mathematicians (including Smale, Anosov, Sinai, and their students) to a *large* class of systems called (uniformly) *hyperbolic diffeomorphisms*.

Even though there is no consensus in the literature about the "perfect" definition of hyperbolic diffeomorphisms, it is widely accepted that the minimal[7] requirement for a diffeomorphism $f : M \to M$ to be called *hyperbolic* is the assumption that its *limit set*[8] (controlling the asymptotic behaviour of all orbits) is a hyperbolic set of f.

In any event, the theory of hyperbolic diffeomorphisms is particularly *beautiful* because it provides a fairly *complete* panorama of the topological and statistical features of systems belonging to certain *open* subsets of the space $\mathrm{Diff}^r(M)$ of C^r-diffeomorphisms of M. For example, it is explained in Bowen's book [12] that if $f : M \to M$ is a hyperbolic C^2-diffeomorphism, then one has

- *finiteness of hyperbolic attractors capturing Lebesgue almost every orbit*: there is a finite collection of hyperbolic sets $\Lambda_1, \dots, \Lambda_k$ of f of the form $\Lambda_j = \overline{\{f^n(q_j) : n \in \mathbb{Z}\}}$, $q_j \in M$, $1 \le j \le k$, whose *basins of attractions*

$$B(\Lambda_j) := \{x \in M : \lim_{n\to+\infty} \mathrm{dist}(f^n(x), \Lambda_j) = 0\}$$

  are open subsets of M such that $\bigcup_{j=1}^{k} B(\Lambda_j)$ has full Lebesgue measure on M;
- *topological dynamics of attractors described by Markov subshifts*: for each $1 \le j \le k$, the restriction $f|_{\Lambda_j}$ of f to Λ_j is topologically semi-conjugated to a *subshift of finite type*[9];
- *statistical description of dynamics via physical measures*: each Λ_j, $1 \le j \le k$, supports an ergodic probability measure μ_j whose basin of attraction

$$B(\mu_j) := \left\{ x \in M : \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi d\mu_j \ \forall \varphi \in C^0(M, \mathbb{R}) \right\}$$

  has positive Lebesgue measure, and, moreover, $\bigcup_{j=1}^{k} B(\mu_j)$ has full Lebesgue measure on M.

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[7]In fact, some authors prefer to work with the following *stronger* notion: a diffeomorphism $f : M \to M$ is hyperbolic when there is a finite collection $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_n = M$ of open sets such that, for each $1 \le j \le n$, one has that $f(\overline{U_j}) \subset U_j$ and the maximal invariant set $\bigcap_{k\in\mathbb{Z}} f^k(U_j \setminus U_{j-1})$ is hyperbolic.

[8]I.e., the closure of the set of accumulation points of all f-orbits.

[9]Namely, the shift dynamics on the space of bi-infinite paths on a strongly connected, directed, finite graph.

Furthermore, it is explained in Shub's book [45] that the subset of hyperbolic C^r-diffeomorphisms is open in $\text{Diff}^r(M)$: in particular, any $g \in \text{Diff}^2(M)$ C^2-close to a hyperbolic diffeomorphism $f \in \text{Diff}^2(M)$ also has a finite number of attractors supporting physical measures whose basins capture Lebesgue almost every orbit.

1.4. **Homoclinic tangencies and Newhouse phenomena.** The marvelous paradigm created by the theory of uniformly hyperbolic diffeomorphisms leads naturally to the question of how frequent are hyperbolic diffeomorphisms in $\text{Diff}^r(M)$.

A celebrated answer to this question was provided by Sheldon Newhouse [31], [32], [33] in the seventies: he showed that the space of hyperbolic diffeomorphisms of a compact *surface* M^2 misses entire open subsets of $\text{Diff}^2(M^2)$.

Before giving the precise statements of some of Newhouse's results, we need to recall some definitions. A *horseshoe* K of a diffeomorphism $f : M \to M$ is an infinite, totally disconnected, hyperbolic set of the form $K = \overline{\{f^n(x) : n \in \mathbb{Z}\}} = \bigcap_{n\in\mathbb{Z}} f^n(U)$ for some $x \in M$ and some neighborhood U of K such that both subbundles E^s and E^u in Definition 1 are non-trivial. A periodic point p belonging to a horseshoe K of a C^2-diffeomorphism $f : M^2 \to M^2$ of a compact surface M^2 displays a *quadratic homoclinic tangency* at $q \in (W^s(p) \cap W^u(p)) \setminus \{p\}$ whenever $W^s(p)$ and $W^u(p)$ meet tangentially with *distinct* curvatures at q.

For later reference, we fix small neighborhoods U of the horseshoe K and V of the homoclinic orbit $\mathcal{O}(q) := \{f^n(q) : n \in \mathbb{Z}\}$ of q as indicated below.

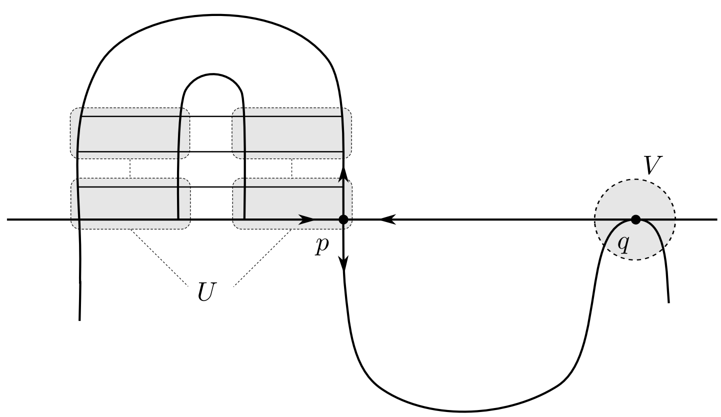

FIGURE 2. Dynamics near a horseshoe exhibiting a quadratic tangency.

If $\mathcal{U}$ is a sufficiently small C^2-neighborhood of f, then all dynamical objects in Figure 2 above admit *continuations*[10]: for any $g \in \mathcal{U}$, the maximal invariant set $K_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is a horseshoe, the periodic point p has a continuation into a nearby periodic point p_g of g, and the compact curves $c^s(f)$ and $c^u(f)$ inside $W^s(p)$ and $W^u(p)$ containing p and q and crossing V have continuations

[10]See, e.g., the book [37] and the references therein for more details.

into a nearby compact curves $c^s(g)$ and $c^u(g)$ in the stable and unstable manifolds of p_g crossing V. This permits to organize the parameter space $\mathcal{U}$ as $\mathcal{U} = \mathcal{U}_- \cup \mathcal{U}_0 \cup \mathcal{U}_+$, where

- $g \in \mathcal{U}_- \iff c^s(g)$ and $c^u(g)$ don't intersect;
- $g \in \mathcal{U}_0 \iff c^s(g)$ and $c^u(g)$ have a quadratic tangency at a point q_g in V;
- $g \in \mathcal{U}_+ \iff c^s(g)$ and $c^u(g)$ have two transverse intersection points in V.

By the implicit function theorem, a quadratic tangency at q means that $\mathcal{U}_0$ is a codimension 1 hypersurface dividing $\mathcal{U}$ into the two connected open subsets $\mathcal{U}_-$ and $\mathcal{U}_+$. The figure below shows the decomposition $\mathcal{U} = \mathcal{U}_- \cup \mathcal{U}_0 \cup \mathcal{U}_+$ of the parameter space and the features on phase space of the elements of $\mathcal{U}_-$, $\mathcal{U}_0$ and $\mathcal{U}_+$.

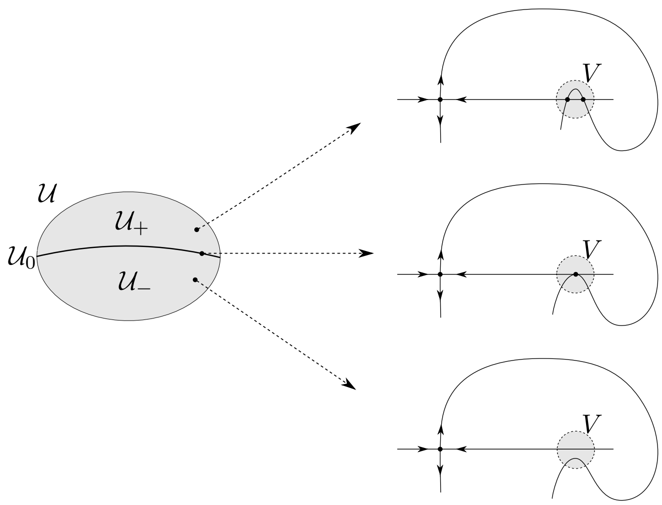

FIGURE 3. Organization of the parameter space $\mathcal{U}$.

In his seminal works, Newhouse used the notion of *thickness*[11] $\tau(C)$ of Cantor sets C of the real line $\mathbb{R}$ to detect some situations where $\mathcal{U}_+$ does *not* contain hyperbolic diffeomorphisms.

Theorem 2 (Newhouse). *Let f be a C^2-diffeomorphism of a compact surface M^2 exhibiting a horseshoe K containing a periodic point p of period $n \in \mathbb{N}$ displaying a quadratic homoclinic tangency as above. Suppose that:*

- *K is thick, i.e., $\tau(W^s(p) \cap K) \cdot \tau(W^u(p) \cap K) > 1$;*
- *f contracts the area near p, i.e., $|det\, df^n(p)| < 1$.*

Then, the subset $R_\infty \subset \mathcal{U}_+$ of diffeomorphisms with infinitely many sinks[12] *is residual in the Baire category sense. In particular, a diffeomorphism in $\mathcal{U}_+$ can not be hyperbolic.*

In the remainder of this subsection, we give a sketch of the proof of this theorem via *persistent tangencies* and a *renormalization* scheme giving sinks related to dissipative periodic points.

[11] See, e.g., Chapter 4 of [37] for the definition and basic properties of the thickness.

[12] Recall that a *sink* is a periodic point s of period m such that all eigenvalues of $df^m(s)$ have moduli < 1.

1.4.1. *Thick horseshoes and persistence of tangencies.* The *generalized stable manifold theorem*[13] asserts that the stable set $W^s(x) := \{y \in M : \text{dist}(f^n(y), f^n(x)) \to 0 \text{ as } n \to +\infty\}$ of any point x in a horseshoe K of a C^2-diffeomorphism $f : M^2 \to M^2$ is an injectively immersed C^2-curve depending continuously on x, that is, the family of stable sets $W^s(x)$, $x \in K$, is a continuous *lamination* with C^2 leaves.

It is known that the laminations above provided by the invariant sets can extended[14] into a stable, resp. unstable *foliation* $\mathcal{F}^s(f)$, resp. $\mathcal{F}^u(f)$ of the neighborhood U of K.

Since $W^s(p)$ and $W^u(p)$ have a quadratic tangency at q, the foliations $\mathcal{F}^s(f)$ and $\mathcal{F}^u(f)$ meet tangentially along a curve $\ell(f)$ containing q: for obvious reasons, $\ell(f)$ is called *line of tangencies.*

If we consider small neighborhoods $W^s_{loc}(p)$ and $W^u_{loc}(p)$ of p in its stable and unstable sets, then the Cantor sets $W^s_{loc}(p) \cap K$ and $W^u_{loc}(p) \cap K$ can be mapped into $\ell = \ell(f)$ via the unstable and stable *holonomies* π^u_f and π^s_f: by definition, $\pi^s_f(x) \in W^s(x) \cap \ell$ and $\pi^u_f(y) \in W^u(y) \cap \ell$ for $x \in W^u_{loc}(p) \cap K$ and $y \in W^s_{loc}(p) \cap K$. Observe that the intersection between the resulting Cantor sets $K^u = \pi^s_f(W^u_{loc}(p) \cap K)$ and $K^s = \pi^u_f(W^s_{loc}(p) \cap K)$ describes all tangencies between the stable and unstable laminations of the horseshoe K near V, that is, by our assumptions, $K^s \cap K^u = \{q\}$.

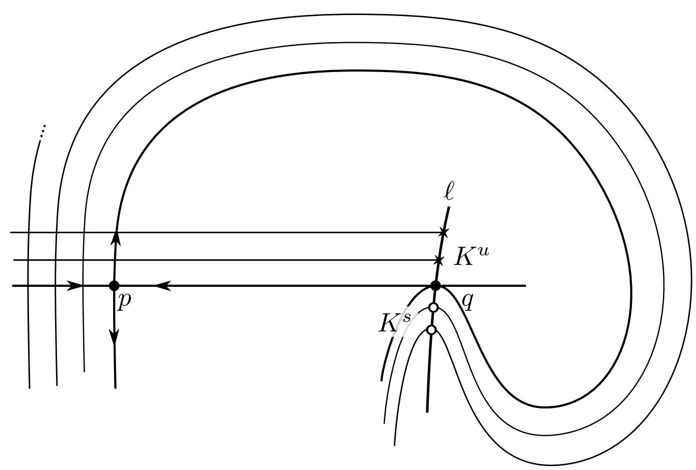

FIGURE 4. The line of tangencies ℓ and the Cantor sets K^s and K^u: the crosses are points in K^u and the dots are points in K^s.

The picture above is perturbed when we consider $g \in \mathcal{U}_+$. Once again, the relevant dynamical objects have continuations: the invariant laminations of K_g extend into stable and unstable foliations $\mathcal{F}^s(g)$ and $\mathcal{F}^u(g)$ giving rise to a line of tangencies $\ell(g)$ containing two Cantor sets $K^s(g) = \pi^u_g(W^s_{loc}(p_g) \cap K_g)$ and $K^u(g) = \pi^s_g(W^u_{loc}(p_g) \cap K_g)$ defined via the unstable and stable holonomies π^u_g and π^s_g. Also, the intersection $K^s(g) \cap K^u(g)$ still accounts for all tangencies between the stable and unstable laminations of the horseshoe K_g.

[13]See Appendix 1 of [37] and references therein (especially [18]).

[14]Such an extension exists by the results of W. de Melo [16] and it heavily depends on the fact that f is a diffeomorphism of a 2-*dimensional* manifold.

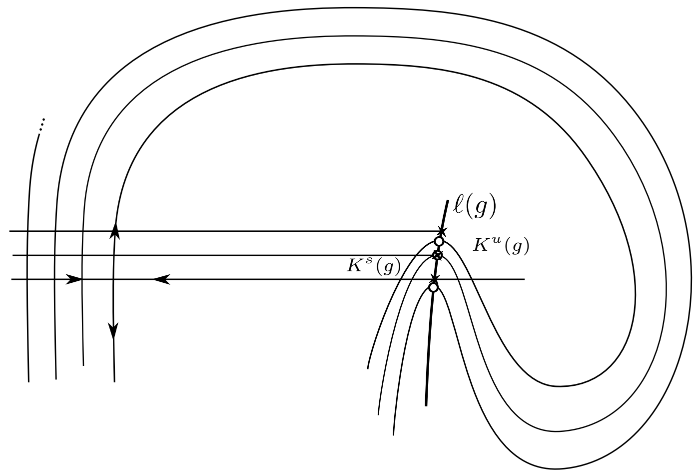

FIGURE 5. The line of tangencies $\ell(g)$ and the Cantor sets $K^s(g)$ and $K^u(g)$ for $g \in \mathcal{U}_+$: the crosses are points in $K^s(g)$ and the dots are points in $K^u(g)$.

Here, Newhouse uses the assumption that K is thick and the fact that the convex hulls $I^s(g)$ and $I^u(g)$ of $K^s(g)$ and $K^u(g)$ in $\ell(g)$ are *linked*[15] when $g \in \mathcal{U}^+$ imply that $K^s(g) \cap K^u(g) \neq \emptyset$ for all $g \in \mathcal{U}_+$. In particular, the quadratic homoclinic tangency at q associated to the periodic point p in the thick horseshoe K of f *persists* when $g \in \mathcal{U}_+$:

Proposition 3. *Under the assumptions of Theorem 2, the stable and unstable laminations of the horseshoe K_g intersect tangentially at some point in V for all $g \in \mathcal{U}_+$.*

1.4.2. *Renormalization, sinks, and conclusion of the proof of Theorem 2.* Recall that we are studying a quadratic homoclinic tangency associated to a periodic point p of period $n \in \mathbb{N}$ such that $|\det df^n(p)| < 1$. By continuity, this implies $|\det dg^n(p_g)| < 1$ for all $g \in \mathcal{U}$. Thus, if $\lambda_g < 1 < \sigma_g$ denote the eigenvalues of $dg^n(p)$, then $|\lambda_g \cdot \sigma_g| = |\det dg^n(p_g)| < 1$.

By Proposition 3, given $g_0 \in \mathcal{U}_+$, we know that the stable and unstable laminations of K_{g_0} meet tangentially at some point in V. Since the stable and unstable manifolds of the periodic point p_{g_0} are dense in the stable and unstable laminations of K_{g_0} (cf. Chapter 2 of [37]), one can apply arbitrarily small perturbations to $g_0 \in \mathcal{U}_+$ so that there is no loss of generality in assuming that $W^s(p_{g_0})$ and $W^u(p_{g_0})$ meet tangentially at some point in the region V. Starting from this quadratic tangency, one gets the following picture for diffeomorphisms $g_\mu \in \mathcal{U}_+$ close to g_0:

[15]I.e., $I^s(g) \cap I^u(g) \neq \emptyset$, but $I^s(g) \not\subset I^u(g)$ and $I^u(g) \not\subset I^s(g)$.

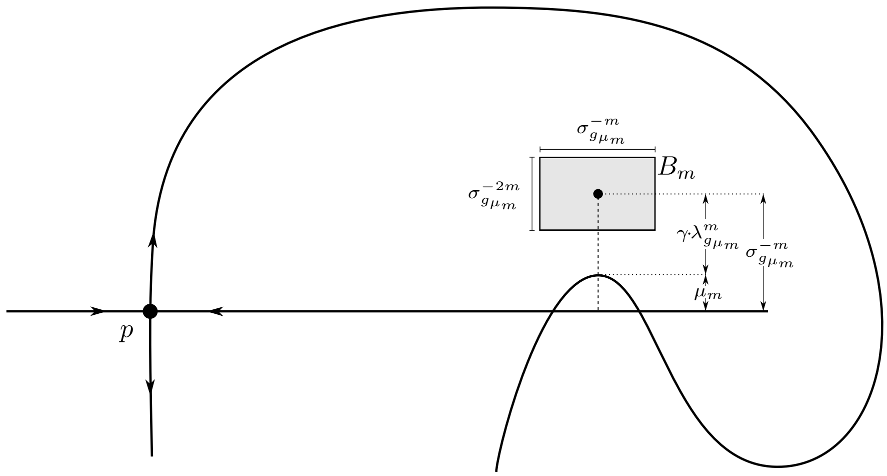

FIGURE 6. Choice of renormalization domain B_m for g_{μ_m}.

As it is shown in Chapter 3 of [37], one can carefully choose parameters[16] μ_m ($m \in \mathbb{N}$) such that

- $\mu_m \to 0$ as $m \to +\infty$ and
- the map $g_{\mu_m}^m|_{B_m}$ can be *renormalized*[17] in such a way that the renormalizations G_m of $g_{\mu_m}^m|_{B_m}$ C^2-converge[18] to the endomorphism $(\widetilde{x}, \widetilde{y}) \mapsto (\widetilde{y}, \widetilde{y}^2)$.

Next, we observe that the endomorphism $(\widetilde{x}, \widetilde{y}) \mapsto (\widetilde{y}, \widetilde{y}^2)$ has an attracting fixed point at $(\widetilde{x}, \widetilde{y}) = (0, 0)$. Therefore, by C^2 convergence of G_m towards this endomorphism, we conclude that $g_{\mu_m}^m$ has an attracting fixed point in B_m for all m sufficiently large. In other words, g_{μ_m} has a *sink* in the region V for all μ_m sufficiently small.

This last statement can be reformulated as follows. For each $m \in \mathbb{N}$, denote by $R_m = \{g \in \mathcal{U}_+ : g \text{ has } m \text{ sinks}\}$. Note that R_m is *open* for all $m \in \mathbb{N}$ (because any sink is persistent under small perturbations of the dynamics). Moreover, since $g_0 \in \mathcal{U}_+$ was arbitrary in the previous argument, we also have that R_1 is *dense* in $\mathcal{U}_+$.

At this stage, the idea of Newhouse is to iterate this argument to show that the set

$$R_\infty = \bigcap_{m \in \mathbb{N}} R_m$$

[16] In principle, the parameters μ must vary in some infinite-dimensional manifold in order to g_μ parametrize a neighborhood of g_0, but for sake of simplicity of the exposition, we will think of this parameter as a *real number* $\mu \in \mathbb{R}$ measuring the distance between the line $W^s(p_{g_\mu}) \cap V$ and the tip of the parabola $W^u(p_{g_\mu}) \cap V$ as indicated in Figure 6.

[17] That is, one can perform an adequate μ_m*-dependent* change of coordinates ϕ_{μ_m} on $g_\mu^m|_{B_m}$ to get a new dynamics $G_m = \phi_{\mu_m}^{-1} \circ g_\mu^m|_{B_m} \circ \phi_{\mu_m}$.

[18] The convergence of the diffeomorphisms G_m towards an endomorphism is *natural* in view of the area-contraction condition $|\det df^n(p)| < 1$: in fact, $g_{\mu_m}^m$ becomes strongly area-contracting as $m \to \infty$ and consequently $g_{\mu_m}^m|_{B_m}$ converges to a curve and G_m converges to an endomorphism of this curve as $m \to \infty$.

of diffeomorphisms of $\mathcal{U}_+$ with *infinitely* many sinks is residual in Baire category sense (and, in particular, R_∞ is *dense* in $\mathcal{U}_+$). Since R_m is open in $\mathcal{U}_+$ for all $m \in \mathbb{N}$ and R_1 is dense in $\mathcal{U}_+$, it suffices to prove that R_{m+1} is dense in R_m for all $m \in \mathbb{N}$ to conclude that R_∞ is residual.

In this direction, one starts with $g_0 \in R_m$ with m periodic sinks $\mathcal{O}_1(g_0), \dots, \mathcal{O}_m(g_0)$. By Proposition 3, we know that the stable and unstable laminations of K_{g_0} meet tangentially somewhere in V. Since $W^s(p_{g_0})$, resp. $W^u(p_{g_0})$, is dense in the stable, resp. unstable, lamination of K_{g_0}, we can assume (up to performing an arbitrarily small perturbation on g_0) that $W^s(p_{g_0})$ and $W^u(p_{g_0})$ meet tangentially at some point $q_{g_0} \in V$ and g_0 has m periodic sinks. Next, we select T a small neighborhood of q_{g_0} such that none of the periodic sinks passes through W, i.e., $W \cap \mathcal{O}_i(g_0) = \emptyset$ for each $i = 1, \dots, m$. By repeating the "renormalization" arguments above (with V replaced by T), one can produce a sequence of diffeomorphisms $(g_{\mu_j})_{j\in\mathbb{N}}$ converging to g_0 as $j \to \infty$ such that g_{μ_j} has a sink $\mathcal{O}(g_{\mu_j})$ passing through T. Because the sinks $\mathcal{O}_i(g_{\mu_j})$ don't pass through T for all j sufficiently large, this means that $\mathcal{O}(g_{\mu_j})$ is a new sink of g_{μ_j}, that is, we obtain that $g_{\mu_j} \in R_{m+1}$ for all j sufficiently large. Since $g_{\mu_j} \to g_0$ as $j \to \infty$, we conclude that R_{m+1} is dense in R_m.

In order to complete the sketch of the proof of Theorem 2, we note that there is no hyperbolic diffeomorphism in $\mathcal{U}_+$: in fact, we saw in Subsection 1.3 that a hyperbolic diffeomorphism $g \in \mathcal{U}_+$ would force the existence of an open subset of $\mathcal{U}_+$ whose elements possess a finite number of hyperbolic attractors, a contradiction with the fact that R_∞ is dense in $\mathcal{U}_+$.

1.5. **Heterodimensional cycles.** The set of hyperbolic diffeomorphisms of a compact 3-manifold M^3 *also* misses entire open subsets of $\text{Diff}^1(M^3)$. In fact, Christian Bonatti and Lorenzo Díaz [8] showed that the existence of open sets $\mathcal{U} \subset \text{Diff}^1(M^3)$ containing a residual subset $\mathcal{R} \subset \mathcal{U}$ such that each $g \in \mathcal{R}$ displays infinitely many sinks. For this sake, they introduced a notion of *persistent connection*[19] between two hyperbolic periodic points p and q of $h \in \text{Diff}^1(M^3)$, and they proved that if $f \in \text{Diff}^1(M^3)$ possesses a persistent connection between two periodic hyperbolic points p and q of periods n and m such that $df^n(p)$ has a *non-real* eigenvalue $|\lambda(p)| < 1$, $df^m(q)$ has a *non-real* eigenvalue $|\sigma(q)| > 1$, and $|\det df^n(p)| < 1$, then there is a neighborhood $\mathcal{U}$ of f containing a residual subset $\mathcal{R} \subset \mathcal{U}$ such that any $g \in \mathcal{R}$ displays infinitely many sinks[20]. Moreover, they noticed that these persistent connections can be constructed with the aid of a special type of horseshoe called *blender* (originally built in [9]).

In a nutshell, a blender is a horseshoe Γ inside a cube $C = [0,1]^3$ whose image under f has the shape in Figure 7 below. In this picture, the local unstable manifolds of the elements of Γ are segments parallel to the x-axis, and the projection of $f(C) \cap C$ along the x-axis to the wall $W = \{0\} \times [0,1]^2$ is the pair of rectangles $R_1 \cup R_2$ in Figure 7 below. In this way, we get an expanding dynamical system $F : R_1 \cup R_2 \to W$ with the property that the *Cantor set*

[19] Namely, there is a subset $\mathcal{D} \subset \text{Diff}^1(M^3)$ such that $\overline{\mathcal{D}}$ is a small neighborhood of h and, for each $g \in \mathcal{D}$, there exists $x_g \in M^3$ with $p_g, q_g \in \overline{\mathcal{O}(x_g)}$.

[20] It is worth to compare this statement with Newhouse's phenomenon in Theorem 2.

$\Omega = \bigcap_{n\in\mathbb{N}} F^{-n}(W)$ intersects[21] any vertical segment $\{0\} \times \{y\} \times [0, 1]$ with y close to 1/2. Note that this property is a sort of "unlikely transversality" because Ω has topological dimension 0 and a curve has topological dimension 1, so that an intersection between these subsets of the space W of topological dimension 2 is "topologically unlikely". In any event, this curious property of Ω extends to the horseshoe Γ in the sense that any C^1-curve of length ≥ 1 which is C^1-close to $\{1/2\} \times \{1/2\} \times [0, 1]$ must intersect $W^u(\Gamma)$. See the original paper [9] and the short expository article [7] for more details.

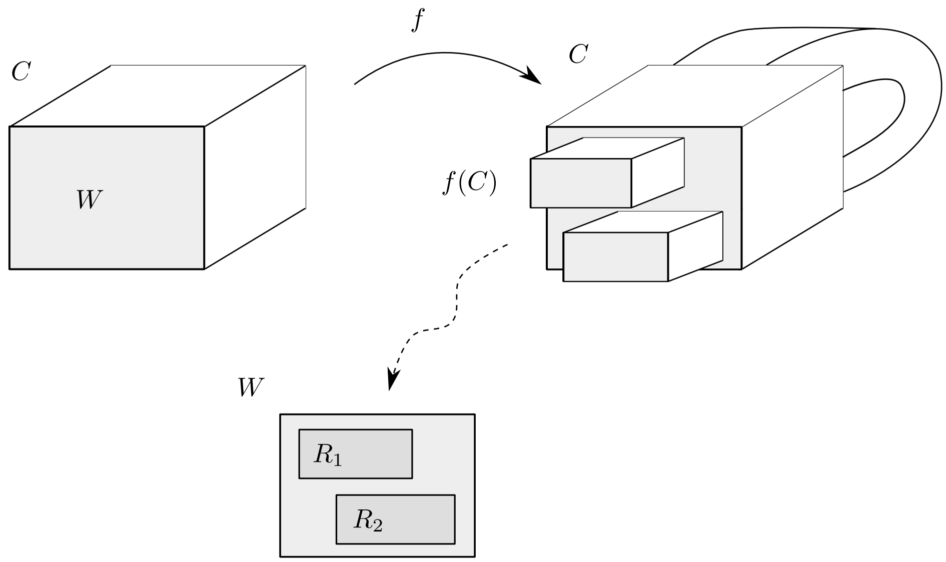

FIGURE 7. Blender horseshoe.

The blender horseshoe Γ allows us to create persistent connections along the following lines. Let $p_1 \in \Gamma$ be a periodic point whose local unstable manifold passes near $\{1/2\} \times \{1/2\} \times [0, 1]$. Suppose that q_1 is a hyperbolic periodic point forming an *heterodimensional cycle* with p_1 in the sense that $\dim(E^u(q_1)) = 2 = \dim(E^s(p_1))$, $\dim(E^s(q_1)) = 1 = \dim(E^u(p_1))$, $W^s(p_1)$ intersects *transversely* $W^u(q_1)$ at some point, and $W^s(q_1)$ contains an almost vertical curve in the cube C intersecting $W^u(p_1)$ (see Figure 8 below). Note that the intersections between $W^s(q_1)$ and $W^u(p_1)$ are *never* transverse (because they are 1-dimensional submanifolds of M^3), so that the heterodimensional cycle involving p_1 and q_1 might seem fragile as any intersection between $W^s(q_1)$ and $W^u(p_1)$ can be easily destroyed. Nonetheless, the key property of the blender horseshoe Γ ensures that, after C^1-small perturbations of the dynamical system, q_1 will form an heterodimensional cycle with some point of Γ (as the unstable lamination $W^u(\Gamma)$ has a persistent intersection with $W^s(q_1)$), and this fact can be exploited to see that p_1 and q_1 have a persistent connection. In particular, this gives plenty of persistent connections since it is not hard to see that any pair of hyperbolic periodic points

[21]This happens basically because there is an interval I containing 1/2 such that, for each $n \in \mathbb{N}$, the vertical projection of $F^{-n}(W)$ contains $\{0\} \times I \times \{0\}$.

p and q such that $W^s(p)$, resp. $W^s(q)$, intersects transversely $W^u(p_1)$, resp. $W^u(q_1)$, and $W^u(p)$, resp. $W^u(q)$, intersects transversely $W^s(p_1)$, resp. $W^s(q_1)$, also has a persistent connection.

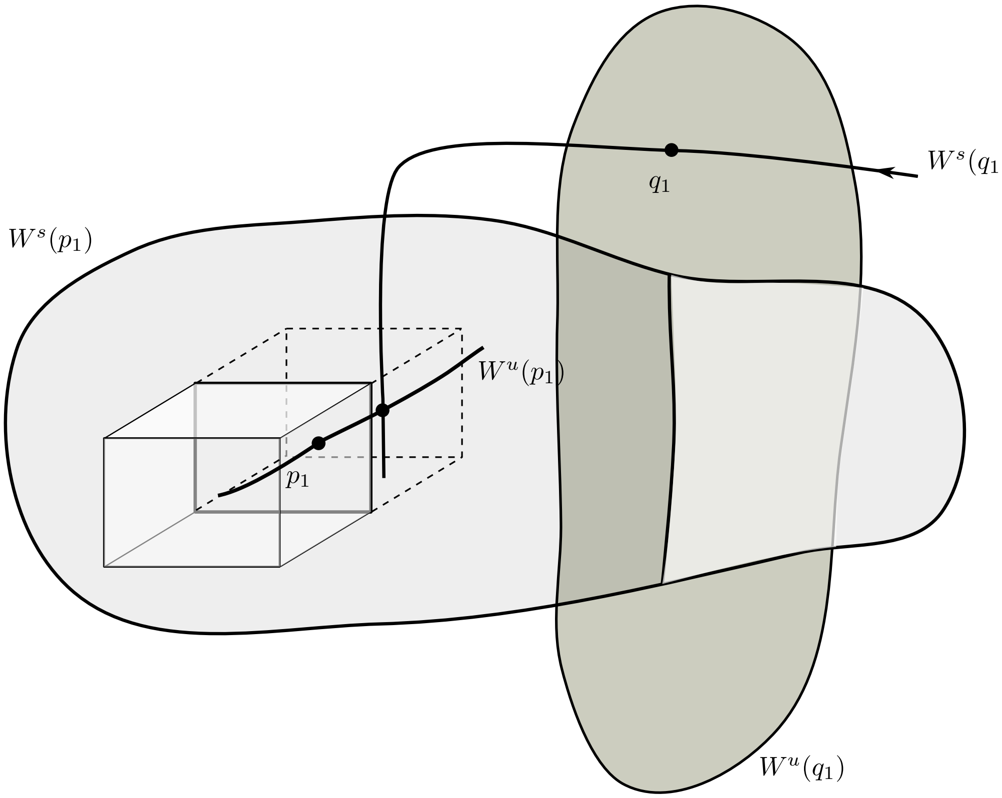

FIGURE 8. Heterodimensional cycle.

1.6. **End of a dream?** We saw in §1.4 and §1.5 above that the bifurcations of certain homoclinic tangencies and heterodimensional cycles lead to open sets of diffeomorphisms which are *not* uniformly hyperbolic. In particular, it is *not* possible to employ the beautiful results in §1.3 to describe the dynamics of *most* orbits of a *dense* set of systems.

On the other hand, Jacob Palis gave a lecture in 1995 (in a conference in honor of Adrien Douady) expressing his view that some of the nice properties enjoyed by uniformly hyperbolic systems should also be true for many dynamical systems. As it turns out, this lecture led to an article [35] where Palis converted his vision into a conjecture whose precise formulation is discussed in the next section.

2. Brief partial review of the first 25 years of history of Palis' conjectures

Let M^n be a compact n-dimensional manifold. We will always assume that M^n has no boundary when $n \geq 2$ and $M^1 = [0,1]$. Given $r \geq 1$, we denote by $D^r(M^n)$ the set of C^r-diffeomorphisms of M^n when $n \geq 2$ and the set of C^r-maps of $[0,1]$ when $n = 1$.

Informally speaking, Palis conjectures that the dynamics of a dense subset of systems in $D^r(M^n)$ can be described in terms of a *finite* number of *attractors* supporting *physical measures* which are *metrically* and *stochastically stable*. In other words, Palis believes that *some* of the features of uniformly hyperbolic systems mentioned in §1.3 above should extend to a dense subset of $D^r(M^n)$.

Logically, it is hard to appreciate the discussion in the previous paragraph without knowing what it is meant by attractor, physical measure, metric and stochastic stability. So, let us now define each one of these notions.

First, it is worth to note that the word "attractor" has several meanings depending on the context: we refer the reader to Milnor's article [30] for a nice account of this topic. For our purposes, an *attractor* A of $f \in D^r(M^n)$ is a subset of the form $A = \overline{\{f^n(x) : n \in \mathbb{Z}\}}$, $x \in M^n$, whose *basin of attraction*

$$B(A) := \{y \in M^n : \lim_{n\to+\infty} \text{dist}(f^n(y), A) = 0\}$$

has positive Lebesgue measure.

Secondly, a *physical measure*[22] μ of $f \in D^r(M^n)$ is a f-invariant supported on an attractor A whose basin of attraction $B(A)$ intersects the basin of attraction of μ

$$B(\mu) := \left\{ z \in M^n : \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \varphi(f^k(z)) = \int \varphi d\mu_j \ \forall\, \varphi \in C^0(M^n, \mathbb{R}) \right\}$$

in a subset of positive Lebesgue measure.

Thirdly, we say that an attractor A of $f \in D^r(M^n)$ is *metrically stable* if for each $k \in \mathbb{N}$ one has that a *generic* k-parameter family $(f_t)_{t\in[-1,1]^k} \subset D^r(M^n)$ with $f_0 = f$ has the property that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for Lebesgue almost every $t \in [-\delta, \delta]^k$ the element $f_t \in D^r(M^n)$ possesses finitely many attractors whose union of basins of attraction coincide with $B(A)$ modulo a subset of Lebesgue measure $< \varepsilon$. In other words, from the point of view of the Lebesgue measure, after "typical" perturbations, nearly all points in the basin of the original attractor will have orbits which are still described by finitely many attractors of the perturbed system.

Finally, we say that an attractor A of $f \in D^r(M^n)$ is *stochastically stable* in its basin of attraction if A carries a physical measure μ such that for any k-parameter family $(f_t)_{t\in[-1,1]^k} \subset D^r(M^n)$ with $f_0 = f$ we have that for each weak-$*$ neighborhood V of μ there exists $\delta > 0$ so that, for Lebesgue almost every choice of $w \in B(A) \cap B(\mu)$ and $(t_1, \dots, t_j, \dots) \in ([-\delta, \delta]^k)^\infty$, the sequence of probability measures

$$\frac{1}{n}\sum_{j=1}^{n} \delta_{f_{t_j}\circ\cdots\circ f_{t_1}(w)}$$

[22]Some authors prefer to say simply that a f-invariant measure μ is physical whenever its basin of attraction $B(\mu)$ has positive Lebesgue measure. However, we will stick to the definition where physical measures are attached to attractors because it is *not* natural to dissociate these objects in the context of Palis' conjectures.

converges in the weak-$*$ topology to an element of V. In other terms, the statistics of the orbits of most points in $B(\mu) \cap B(A)$ under "typical" small random perturbations of f is still described by a probability measure close to μ.

2.1. **Formal statement of Palis' global conjecture.** The global (main) conjecture by Palis asserts that there exists a dense subset $D \subset D^r(M^n)$ such that for any $f \in D$ one has the following four properties:

(i) f possesses finitely many attractors whose basins of attraction cover a subset of M^n with full Lebesgue measure.
(ii) each attractor of f supports a physical measure.
(iii) each attractor of f is metrically stable.
(iv) each attractor of f is stochastically stable.

Moreover, if $n = 1$ (i.e., for one-dimensional systems), then Palis says that we also have that

(v) for a generic k-parameter family $(f_t)_{t\in[-1,1]^k} \subset D^r(M^1)$, the attractors of f_t are regular (i.e., periodic sinks) or stochastic (i.e., carry an invariant probability measure which is absolutely continuous with respect to the Lebesgue measure on M^1) for Lebesgue almost every $t \in [-1, 1]^k$.

In summary, Palis conjectures that the dynamics of *Lebesgue almost all* orbits of *many* systems are described by *finitely many* attractors which are *probabilistically stable*.

Remark 4. The statement above can be found at the end of the first section of the article [35] and at Subsection 2.7 of the paper [36]. Also, the reader will find in these references that Palis believes that similar statements should be true for continuous-time systems (i.e., flows and vector-fields), but, for the sake of simplicity of exposition, we will stick to the discrete-time cases here.

2.2. **Palis' program: reduction of global conjecture to analysis of certain bifurcations.** Besides proposing his global (main) conjecture, Palis suggested that its solution could be found in the detailed study of the bifurcations arising from the perturbations of homoclinic tangencies and heterodimensional cycles (cf. the second section of [35] and the Subsection 3.3 of [36]).

More concretely, Palis thinks that the two phenomena described in §1.4 and §1.5 above are the sole obstructions for the denseness of uniformly hyperbolic systems in $D^r(M^n)$ when $n \geq 2$: the subset B of $f \in D^r(M^n)$, $n \geq 2$, exhibiting a homoclinic tangency[23] or a heterodimensional cycle[24] is dense in the complement of the closure of the subset of uniformly hyperbolic systems in $D^r(M^n)$.

If the statement in the previous paragraph is correct, then one could try to establish Palis' global conjecture by showing that the bifurcations of homoclinic tangencies and heterodimensional cycles contain systems satisfying items (i) to (v) above, i.e., by perturbing / unfolding homoclinic tangencies and heterodimensional cycles to get the conclusions of Palis' global conjecture.

[23] I.e., a hyperbolic periodic point whose invariant manifolds meet tangencially at some point.

[24] I.e., a finite collection of hyperbolic periodic points $p_1, \dots, p_m$ such that their stable manifolds have different dimensions and $W^s(p_{i+1}) \cap W^u(p_i) \neq \emptyset$ for all $i = 1, \dots, m$ (with the convention that $p_{m+1} = p_1$).

2.3. **Some partial results towards Palis' global conjecture.** As the reader might suspect, it is impossible to review in a short survey the huge literature developed around Palis' global conjecture in the last 25 years. For this reason, we chose to briefly discuss below two low-dimensional settings where striking progresses were made.

2.3.1. *Palis' global conjecture for one-dimensional dynamical systems.* Since the early stages of the theory of one-dimensional real and complex maps, it was quickly realized that the behaviour of the orbits of the *critical* points have a great influence on the global dynamics of the map: for instance, Fatou proved that a rational map of the Riemann sphere with degree $d \geq 2$ is uniformly hyperbolic if and only if the orbit of each critical point converges to a periodic sink.

Therefore, it is natural to start the quest of understanding the dynamics of one-dimensional maps by looking at those with a *single* critical point. It is clear that such maps can be found among polynomials of degree 2 and, in a certain sense, the *quadratic family* $f_c(x) = x^2 + c$ provides the most emblematic examples of unimodal maps.

The quadratic family hides an amazing amount of distinct dynamical behaviours in its simple appearance. For example, Jakobson [19] showed that the set of parameters $c \in [-2, 0)$ such that f_c is stochastic (i.e., preserves an absolutely continuous invariant probability measure) has positive Lebesgue measure, and Lyubich [23] and Graczyk and Światek [17] independently established that the set of parameters $c \in [-2, 1/4]$ such that f_c is regular (i.e., possesses a periodic sink) is open and dense in $[-2, 1/4]$. Since it is not hard to check that the regular and stochastic behaviours can't coexist, we conclude that the regular and stochastic parameters $c \in [-2, 1/4]$ pop up in a complicated way inside $[-2, 1/4]$.

After that, Lyubich [24] famously established in 2002 that the union of the subsets of regular and stochastic parameters in $c \in [-2, 1/4]$ has full Lebesgue measure in $[-2, 1/4]$, so that *the fifth item of Palis' global conjecture is true for the quadratic family.*

The proofs of the results in the previous paragraphs are quite long, but, as we hinted above, their hearts lie in the careful analysis of the orbit of the critical point at $x = 0$: for example, the regular parameters arise when the orbit of the critical point converges to a periodic sink and the stochastic parameters often arise when the orbit of the critical point is slowly recurrent[25].

Subsequently, Avila and Moreira [4] proved in 2003 that the conclusion of the fifth item of Palis' global conjecture is also true for generic families of the so-called *S-unimodal maps.* On the other hand, Palis' global conjecture is still not fully solved for generic families of one-dimensional systems (despite the efforts of several authors including Bruin, Rivera-Letelier, Shen and van Strien [13]).

[25]For example, Benedicks and Carleson [6] found many stochastic parameters by performing an *exclusion of parameters c* failing the *Collet–Eckmann condition* $|(f_c^n)'(f_c(0))| \geq \exp(n^{2/3})$ for all large n (and this condition should be thought as a sort of slow recurrence condition on the orbit of the critical point 0 because $(f_c^n)'(f_c(0)) = \prod_{j=0}^{n-1} f_c'(f_c^j(c))$ and $|f_c'(x)| = 2|x|$ is proportional to the distance between x and 0).

2.3.2. *Physical measures beyond uniform hyperbolicity.* The notion of uniform hyperbolicity was generalized by several authors in order to describe the dynamics of large classes of systems. Among these concepts, the theories of *partially hyperbolic* systems and *non-uniformly hyperbolic* systems saw huge developments in the last three decades. By employing several techniques from these theories, Alves, Bonatti and Viana were able to prove in 2000 that a robust class of partially hyperbolic attractors with definite non-uniformly hyperbolic behaviours support physical measures and, hence, they satisfy the *second item* of Palis' global conjecture.

More precisely, Bonatti and Viana [10] considered $f \in D^r(M^n)$, $r \geq 2$, $n \geq 3$, possessing an attractor $\Lambda = \bigcap_{n\in\mathbb{N}} f^n(U)$, where U is an open set with $f(\overline{U}) \subset U$, which is *partially hyperbolic* and *mostly contracting* in the sense that there is a df-invariant decomposition $T_\Lambda M = E^{cs} \oplus E^u$ and some constant $0 < \lambda < 1$ such that

- E^u is uniformly expanding and the largest expansion[26] along E^{cs} is dominated by the weakest expansion along E^u, i.e., $\|df^{-1}|_{E^u(x)}\| \leq \lambda$ and $\|df|_{E^{cs}(x)}\| \cdot \|df^{-1}|_{E^u(f(x))}\| \leq \lambda$ for all $x \in \Lambda$;
- the subbundle E^{cs} is mostly contracting, i.e., for any disk D^u contained in some unstable manifold $W^u(y)$, $y \in \Lambda$, one has $\lambda^{cs}(x) := \limsup_{n\to\infty} \frac{1}{n} \log \|df^n|_{E^{cs}(x)}\| < 0$ for a subset of $x \in D^u$ with positive Lebesgue measure.

In this context, Bonatti and Viana showed that the second item of Palis' global conjecture holds for the attractor Λ of f because, up to a subset of zero Lebesgue measure, the basin $B(\Lambda) = \bigcap_{n\in\mathbb{N}} f^{-n}(U)$ of attraction of Λ coincides with the union of the basins $B(\mu_1), \dots, B(\mu_k)$ of finitely many physical measures $\mu_1, \dots, \mu_k$ supported on Λ.

Very roughly speaking, Bonatti and Viana obtained the result in the previous paragraph (which covers certain open subsets of $D^r(M^n)$ containing no uniformly hyperbolic system) via the following construction. Let m_{D^u} be the normalized Lebesgue measure on a disk contained in some unstable manifold of a point in Λ. The work of Pesin and Sinai [40] ensures that the accumulation points of the Cesàro averages $\frac{1}{n}\sum_{j=0}^{n-1} f_*^j(m_{D^u})$ of the push-forwards of m_{D^u} under the iterates of f produce *u-Gibbs states*, i.e., f-invariant probability measures whose conditional measures along unstable manifolds are absolutely continuous with respect to Lebesgue. Furthermore, Pesin's theory of non-uniformly hyperbolic systems can be used to show that u-Gibbs states are *physical measures when E^{cs} is mostly contracting* (cf Remark 2.5 in [10]). At this point, Bonatti and Viana develop these facts to prove that any u-Gibbs state leads to ergodic physical measures with *large*, mutually disjoint basins covering $B(\Lambda)$ modulo a subset of zero Lebesgue measure, so that Λ supports a finite number of physical measures whose basins capture Lebesgue almost every point in $B(\Lambda)$.

Afterwards, Alves, Bonatti and Viana [2] considered $f \in D^r(M^n)$, $r \geq 2$, $n \geq 3$, possessing a compact subset K which is positively invariant (i.e., $f(K) \subset K$), partially hyperbolic and *mostly*

[26] In comparison with the notion of uniform hyperbolicity in Definition 1, we do not request uniform contraction along E^{cs} (and, actually, E^{cs} is allowed to exhibit some weak expansion).

expanding in the sense that there is a df-invariant decomposition $T_K M = E^s \oplus E^{cu}$ and some constant $0 < \lambda < 1$ such that

- E^s is uniformly contracting and the smallest contraction along E^{cu} dominates the weakest contraction along E^s, i.e., $\|df|_{E^s(x)}\| \leq \lambda$ and $\|df|_{E^s(x)}\| \cdot \|df^{-1}|_{E^{cu}(f(x))}\| \leq \lambda$ for all $x \in K$;
- the subbundle E^{cu} is mostly expanding, i.e., $\limsup\limits_{n\to\infty} \frac{1}{n} \sum\limits_{j=1}^{n} \log \|df^{-1}|_{E^{cu}(f^j(x))}\| < 0$ for a subset of $x \in K$ with positive Lebesgue measure.

In this setting, Alves, Bonatti and Viana proved that $\Lambda = \bigcap\limits_{n\in\mathbb{N}} f^n(K)$ supports a physical measure (and, hence, Λ satisfies the second item of Palis' global conjecture). As it turns out, the "mostly expanding" situation is more difficult than the "mostly contracting" case because the subbundle E^{cu} does *not* have the nice uniform expansion properties required by Pesin and Sinai to build u-Gibbs states eventually leading to physical measures. In order to overcome this problem, Alves, Bonatti and Viana noticed that we still can produce physical measures by looking at the accumulation points of $\frac{1}{n} \sum\limits_{j=0}^{n-1} (f^j)_*(m_D)$, where m_D is the normalized Lebesgue measure on a disk D almost tangent to E^{cu}, because the mostly expanding condition on E^{cu} permits to get *uniform expansion along a sequence with positive frequency of hyperbolic times* thanks to the so-called *Pliss lemma*[27].

Remark 5. In 2007, Alves, Araújo and Vásquez [1] exhibited open subsets of mostly expanding partially hyperbolic diffeomorphisms f which are *weakly* stochastically stable (i.e., they satisfy a *weak* form of the fourth item of Palis' global conjecture) in the sense that the accumulation points of Cesàro averages of Dirac masses along the orbits of random perturbations of f tend to convex combinations of the physical measures of f when the size of the random noise goes to zero.

Remark 6. In 2018, Andersson and Vásquez [3] proposed a *variant* of the notion of mostly expanding partially hyperbolic diffeomorphism, and they showed that their concept of mostly expanding *always* lead to open subsets of $D^r(M^n)$, $r > 1$ whose elements possess a finite number of physical measures whose basins cover Lebesgue almost every point of M^n. In particular, mostly expanding partially hyperbolic diffeomorphisms (in Andersson–Vásquez sense) form an open class of systems verifying the *analog* of the first and third items of Palis' global conjecture where "attractors" are replaced by "physical measures".

After the works of Alves, Bonatti and Viana, it is natural to ask whether the construction of physical measures can be performed when the mostly contracting and/or the mostly expanding conditions fail. In this direction, Tsujii [47] made a breakthrough in 2005 by establishing a version of the first, second and third items of Palis' global conjecture for partially hyperbolic *endomorphisms*

[27] This lemma ensures that if $c > 0$ is a constant such that $\sum\limits_{j=1}^{n} -\log \|df^{-1}|_{E^{cu}(f^j(x))}\| > 2c\cdot n$ for all n sufficiently large, then there is a subset $T \subset \mathbb{N}$ with positive density consisting of "uniformly hyperbolic times" in the sense that $\sum\limits_{j=k}^{t} -\log \|df^{-1}|_{E^{cu}(f^j(x))}\| > c(t-k)$ for all $t \in T$ and $0 \leq k < t$.

of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. More concretely, let $PH^r(\mathbb{T}^2)$ be the subset of C^r partially hyperbolic endomorphisms of $\mathbb{T}^2$, i.e., C^r-maps $f : \mathbb{T}^2 \to \mathbb{T}^2$ such that there exists a splitting $T\mathbb{T}^2 = E^c \oplus E^u$ into 1-dimensional subbundles and some constants $C > 0$, $\mu > 1$ with the following properties:

- E^u is uniformly expanding, i.e., $\|df^n|_{E^u}\| \geq C\mu^n$ for all $n \geq 0$;
- the strongest expansion along E^c is dominated by the weakest expansion along E^u, i.e., $\|df^n|_{E^c}\| \leq C^{-1}\mu^{-n}\|df^n|_{E^u}\|$ for all $n \geq 0$.

It is not difficult to check that $PH^r(\mathbb{T}^2)$ is an open subset of the space $C^r(\mathbb{T}^2, \mathbb{T}^2)$ of smooth self-maps of $\mathbb{T}^2$. Tsujii considered the subset $\mathcal{G}^r(\mathbb{T}^2)$ of $f \in PH^r(\mathbb{T}^2)$ possessing a finite number of ergodic physical measures whose union of basins have total Lebesgue measure on $\mathbb{T}^2$ and he established a version of the first and second items of Palis' global conjecture for $PH^r(\mathbb{T}^2)$ by proving that $\mathcal{G}^r(\mathbb{T}^2)$ is residual whenever $r \geq 19$. Moreover, Tsujii obtained a version of the third item of Palis' global conjecture for $PH^r(\mathbb{T}^2)$ by showing that, for a generic family $(f_t)_{t\in[-1,1]^k} \subset PH^r(\mathbb{T}^2)$, one has that $f_t \in \mathcal{G}^r(\mathbb{T}^2)$ for Lebesgue almost every $t \in [-1, 1]^k$.

In a certain sense, the proof of Tsujii's theorems above are divided into two regimes depending on the behaviour of the central subbundle E^c of $f \in PH^r(\mathbb{T}^2)$. If E^c is "mostly contracting" or "mostly expanding", then the ideas of Alves, Bonatti and Viana can be used to get the desired results. On the other hand, if E^c is "neutral", then Tsujii obtains his results by proving that certain *transversality conditions* ensure that the Pesin–Sinai method of construction of u-Gibbs states (i.e., taking accumulation points of $\frac{1}{n}\sum\limits_{j=0}^{n-1}(f^j)_*(m_\gamma)$ where m_γ is the normalized Lebesgue measure on a *curve* γ almost tangent to E^u) leads to invariant probability measures which are absolutely continuous with respect to the Lebesgue measure $m_{\mathbb{T}^2}$ of the 2-*dimensional* torus $\mathbb{T}^2$ and, *a fortiori*, physical measures. In a nutshell, the transversality conditions of Tsujii are setup in order to force the iterates $f^n(\gamma)$ of γ to "almost fill" open subsets as $n \to \infty$ like it is indicated in Figure 9 below. In this way, the iterates f^n of f "spread" the one-dimensional density m_γ over open subsets by making it converge to a non-trivial density function on the 2-dimensional phase space $\mathbb{T}^2$.

Remark 7. In 2019, Bortolotti [11] extended the scope of Tsujii's results to include *certain* open sets of C^r-*diffeomorphisms* of 3-manifolds, $r \geq 2$, possessing partially hyperbolic attractors with neutral central direction and Lipschitz stable lamination.

2.4. **Some partial results towards Palis' program.** Similarly to the case of Palis' global conjecture, we dispose of an important literature providing partial progress towards Palis' program. Hence, we will not try to pursue here the virtually impossible task of reviewing the whole body of articles in this topic, but rather we shall focus in the sequel on three outstanding theorems.

2.4.1. *The first part of Palis' program for C^1-diffeomorphisms.* In 2000, Pujals and Sambarino [44] completed the first part of Palis' program for C^1-diffeomorphisms of compact surfaces M^2 by

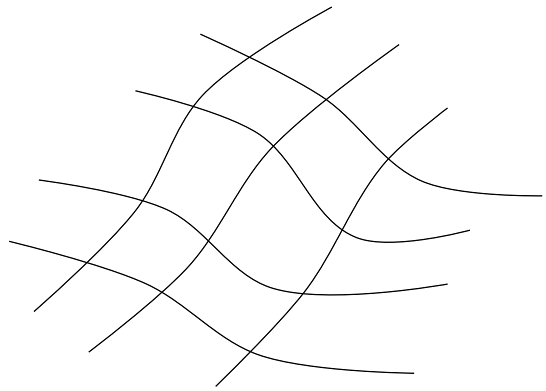

FIGURE 9. Tsujii's transversality condition on almost unstable curves.

showing that any $f \in D^1(M^2)$ can be C^1-approximated by an uniformly hyperbolic diffeomorphism or by a diffeomorphism displaying a homoclinic tangency.

The proof of Pujals–Sambarino theorem starts with some fundamental ideas of Mañé [25] on his famous solution of the C^1-stability conjecture asserting that uniformly hyperbolic systems are characterised by the stability of their phase portraits under small C^1-perturbations.

More concretely, let $\mathcal{U}^1$ be the open subset of $f \in D^1(M^2)$ which is *far* from homoclinic tangencies. In this context, our goal is to show that the elements of $\mathcal{U}^1$ are C^1-approximated by uniformly hyperbolic diffeomorphisms. For this sake, one begins by proving that a generic element $f \in \mathcal{U}^1$ possesses only hyperbolic periodic points whose stable and unstable subspaces have an angle *uniformly* bounded away from zero. Since the periodic points of a generic element $f \in D^1(M^2)$ are dense in its non-wandering[28] set $\Omega(f)$ (thanks to the celebrated Pugh's C^1-closing lemma [43]), we can use the splittings of these (hyperbolic) periodic points to build a *dominated splitting* over $\Omega(f)$, i.e., a df-invariant splitting $T_{\Omega(f)}M^2 = E \oplus F$ such that, for some constants $C > 0$ and $0 < \lambda < 1$, one has

$$\|df^n(x)|_E\| \cdot \|df^{-n}(f^n(x))|_F\| \leq C\lambda^n$$

for all $x \in \Omega(f)$ and $n \in \mathbb{N}$ (or, in plain terms, the largest expansion along E is dominated by the weakest contraction along F). Therefore, we reduced the task of deriving the uniform hyperbolicity of a generic element $f \in \mathcal{U}^1$ to show that a dominated splitting $E \oplus F$ over $\Omega(f)$ is uniformly hyperbolic.

At this point, Mañé proved in the *context of the C^1-stability conjecture* that the dominated splitting $E \oplus F$ over $\Omega(f)$ *must* be uniformly hyperbolic, i.e., of the form $E^s \oplus E^u$ with $\|df^n|_{E^s}\| \leq C\lambda^n$ and $\|df^{-n}(x)|_{E^u}\| \leq C\lambda^n$ for all $x \in \Omega(f)$ and $n \in \mathbb{N}$, because a dominated non-hyperbolic

[28] I.e., the set of points $x \in M^2$ such that for each neighborhood $U \in x$ there exists $n > 0$ such that $f^n(U) \cap U \neq \emptyset$.

splitting would allow to construct *non-hyperbolic* periodic points for some C^1-perturbations of f and this fact can easily be used to show that the phase portrait of f is not stable under small C^1-perturbations.

In the setting of Pujals–Sambarino, we are no longer assuming that f is structurally stable, but merely that f is a generic element of $\mathcal{U}^1$. In particular, the coexistence of dominated splitting and non-hyperbolic periodic points is now allowed and we have to use again our assumption that $f \in \mathcal{U}^1$ is far from homoclinic tangencies to analyse the dominated splitting $E \oplus F$. Here, Pujals and Sambarino take inspiration from another work of Mañé [26] where it is shown that if Λ is a compact invariant subset of a C^2-endomorphism $h : N \to N$, $N = [0,1]$ or S^1, such that Λ contains no critical point of h and all periodic points in Λ are sources, then either Λ is an uniformly expanding set of h or ($N = S^1$ and) h is topologically conjugated to an irrational circle rotation. More precisely, they "reinterpreted" the dominated splitting condition as a sort of "absence of dynamically critical points" in order to vastly generalise Mañé's ideas about one-dimensional endomorphisms to get the following result. Let $\Lambda \subset \Omega(f)$ be a compact invariant subset of the non-wandering set of a C^2-diffeomorphism $g \in D^2(M^2)$. Suppose that Λ admits a dominated splitting $T_\Lambda M^2 = E \oplus F$ and all periodic points in Λ are hyperbolic of saddle type. Then, $\Lambda = H \cup C_1 \cup \dots \cup C_k$, where H is a hyperbolic set of g, C_i, $i = 1, \dots, k$, are closed curves which are g^{m_i}-periodic for some $m_i \in \mathbb{N}$ and *normally hyperbolic*[29], and each $g^{m_i}|_{C_i} : C_i \to C_i$ is topologically conjugated to an irrational rotation on a circle S^1.

Since the closed curves supporting topological irrational rotations can be easily destroyed by perturbations, the statement from the previous paragraph could be used by Pujals and Sambarino to conclude that a generic element $f \in \mathcal{U}^1$ is uniformly hyperbolic, as desired.

In 2015, Crovisier and Pujals [15] established a *topological version* of the first part of Palis' program for C^1-diffeomorphisms of higher-dimensional compact manifolds M^n, $n \geq 3$, by showing that any $f \in D^1(M^n)$ can be approximated by a diffeomorphism displaying a homoclinic tangency, a heterodimensional cycle or which is *essentially uniformly hyperbolic* in the sense that there is a finite number of hyperbolic attractors whoses basins cover an *open and dense* subset of M^n.

2.4.2. *Second part of Palis' program for C^∞ surface diffeomorphisms.* In the direction of trying to prove Palis' global conjecture for smooth surface diffeomorphisms, it is natural to investigate how *frequent* is the Newhouse phenomenon of locally generic coexistence of infinitely many sinks and sources after the unfolding of tangencies along generic one-parameter families.

In 1987, Newhouse, Palis and Takens considered again the context of §1.4, that is, a smooth surface diffeomorphism $f : M \to M$ possessing a horseshoe $K = \bigcap_{n\in\mathbb{Z}} f^n(U)$ containing a periodic point p displaying a quadratic homoclinic tangency at $q \in V$. They proved in [34] and [38] that if K has Hausdorff dimension *strictly less* than one and $(g_t)_{t\in[-1,1]}$ is a generic one-parameter family with $g_0 = f$, then $\Lambda_{g_t} := \bigcap_{n\in\mathbb{Z}} g_t^n(U \cup V)$ is a (uniformly) hyperbolic horseshoe for *most* t near 0 in

[29] I.e., the restriction of $E \oplus F$ to C_i has the form $E^s \oplus TC_i$ or $TC_i \oplus E^u$.

the sense that

$$\lim_{\varepsilon\to 0}\frac{\text{Leb}(\{|t|<\varepsilon:\Lambda_{g_t}\text{ is not a horseshoe}\})}{2\varepsilon}=0.$$

Intuitively, this result is explained by the ideas that Λ_{g_t} is a horseshoe when the stable and unstable Cantor sets $K^s(g_t)$ and $K^u(g_t)$ defined in §1.4.1 do not intersect, $K^s(g_t)$ and $K^u(g_t)$ are close to $K^s(f)+t$ and $K^u(f)$ for $|t|$ small, and the set of parameters t such that $K^s(f)+t$ and $K^u(f)$ intersect has zero Lebesgue measure because $(K^s(f)+t)\cap K^u(f)\neq\emptyset$ if and only if t belongs to the *arithmetic difference* $K^u(f)-K^s(f)$ which has zero Lebesgue measure since its Hausdorff dimension is strictly less than one thanks to our assumption on K.[30]

On the other hand, Newhouse phenomenon indicates that we can not expect Λ_{g_t} to be a (uniformly) hyperbolic horseshoe for *all* t when K has Hausdorff dimension larger than one. Nevertheless, Palis and Yoccoz [39] made a *tour-de-force* in 2009 in a long work (of 217 pages) by showing that we can still expect Λ_{g_t} to be a *non-uniformly hyperbolic horseshoe* for most t near 0.

Even though the precise definition of a non-uniformly hyperbolic horseshoe $\Lambda_{g_{t_0}}$ is very technical, the reader must keep in mind that they are "saddle-type objects" in the sense that their local stable and unstable sets

$$W^s(\Lambda_{g_{t_0}}):=\bigcap_{n\leq 0}g_{t_0}^n(U\cup V)\quad\text{and}\quad W^u(\Lambda_{g_{t_0}}):=\bigcap_{n\geq 0}g_{t_0}^n(U\cup V)$$

have zero Lebesgue measure on M^2. In particular, such a $\Lambda_{g_{t_0}}$ can not carry sinks or sources, and, hence, the presence of a non-uniformly hyperbolic horseshoe for g_{t_0} *prevents* the Newhouse phenomenon on $U\cup V$ for the parameter t_0.

Anyhow, the concrete situation considered by Palis and Yoccoz in [39] was the following. We have a smooth surface diffeomorphism f possessing a horseshoe K containing two periodic points p_s and p_u lying in distinct orbits such that $W^s(p_s)$ and $W^u(p_u)$ have a first[31] quadratic tangency at a point q . If the stable and unstable Cantor sets K^s and K^u of K have Hausdorff dimensions d^s and d^u satisfying[32]

$$(d^s+d^u)^2+\max\{d^s,d^u\}^2<(d^s+d^u)+\max\{d^s,d^u\},$$

then a generic one-parameter family $(g_t)_{|t|\leq 1}$ with $g_0=f$ displays a non-uniformly hyperbolic horseshoe Λ_{g_t} near $K\cup\mathcal{O}(q)$ for most t near zero in the sense that

$$\lim_{\varepsilon\to 0}\frac{\text{Leb}(\{|t|<\varepsilon:\Lambda_{g_t}\text{ is a non-uniformly hyperbolic horseshoe}\})}{2\varepsilon}=1.$$

Closing this text, let us say a few words about the general idea behind Palis–Yoccoz work [39]. By taking inspiration on Yoccoz proof of Jakobson's theorem [49], one tries to define a notion

[30] As the dimension of $K^u(f)-K^s(f)$ is at most the Hausdorff dimension of the product set $K^u(f)\times K^s(f)$ whose dimension coincides with the dimension of K.

[31] In the sense that there are neighborhoods U of K and V of q such that $\bigcap_{n\in\mathbb{Z}}f^n(U\cup V)$ is reduced to the union of K and the orbit of q.

[32] Note that this condition permits to consider certain horseshoes K whose Hausdorff dimension d^s+d^u is slightly larger than one.

of *strongly regular parameter* t detecting the non-uniform hyperbolicity of Λ_{g_t}. For this sake, one introduces some coordinates on $U \cup V$ so that (some fixed powers of) g is an *affine-like*[33] hyperbolic map on U and a folding ("Hénon-like") map on V. In this context, Palis and Yoccoz propose to capture the non-uniformly hyperbolic points in Λ_{g_t} using decreasing sequences of domains and images of certain classes of affine-like hyperbolic *iterates* of g_t. Here, there is no guarantee that the subset of points captured by this scheme cover a significant portion of Λ_{g_t}, but Palis and Yoccoz noticed that this is the case whenever the number of *bi-critical*[34] affine-like iterates of g is relatively small. At this stage, Palis and Yoccoz complete the proof of their theorem by performing an *exclusion of parameters* (based on their assumption on d^s and d^u) to ensure that g_t has few bi-critical affine-like iterates for most t near zero.

Remark 8. Besides consulting the original article [39], the curious reader can find more informations about non-uniformly hyperbolic horseshoes on the texts [27], [28] and [29].

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[33] In the sense that the dynamics maps an almost vertical strip to almost horizontal strip by expanding almost horizontal directions and contracting almost vertical directions.

[34] An affine-like iterate of g is called bi-critical when its domain and image are strips passing too close to the tip of the parabolas associated to the unfolding of tangencies.

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